幾何学における ラリター・シュインガー場

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Laplace op **on $\mathbb{R}^n$** $\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$

on cpt Riem $(M.g)$ $\frac{\partial}{\partial x_i} \rightsquigarrow \nabla_{e_i} = \frac{\partial}{\partial x_i} + \alpha_i$ $\{e_i\}$ o.n.f.

$\boxed{\Delta_p := d\delta + \delta d = \nabla^*\nabla + R_p}$ on $\Omega^p(M)$, $(R_1 = Ric)$

Weitzenböck

- harmonic p-form

$$\mathbb{H}^p(M) := \{\varphi \in \Omega^p(M) \mid \Delta_p \varphi = 0\} \quad \dim = b_p(M)$$

$$\cong H^p(M, \mathbb{R})$$

Hodge-deRham

$p=1$, $Ric > 0 \Rightarrow b_1(M) = 0$ vanishing thm

- $(M.g)$ + 幾何構造 $\mathbb{H}^p(M) = \cdots$, 色々な vanishing thm

Dirac operator

On $(\mathbb{R}^2, g_0)$, $e_1 \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $e_2 \mapsto \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$D := e_1 \partial/\partial x_1 + e_2 \partial/\partial x_2 = 2\begin{pmatrix} 0 & \partial/\partial z \\ \partial/\partial \bar{z} & 0 \end{pmatrix} : C^\infty(\mathbb{R}^2, \mathbb{C}^2) \to C^\infty(\mathbb{R}^2, \mathbb{C}^2)$$

$e_1^2 = e_2^2 = -1$, $e_1 e_2 = -e_2 e_1$ より.

$$\boxed{D^2 = \Delta}$$

On $(\mathbb{R}^3, g_0)$, $\sqrt{-1}\, e_i \longmapsto \sigma_i$ Pauli 行列

$$D = \sum_{i=1}^{3} e_i \frac{\partial}{\partial x_i},$$

$e_i e_j + e_j e_i = -2\delta_{ij}$

$\rightsquigarrow \boxed{D^2 = \Delta}$

On (M, g),

$$C^\infty(M, \mathbb{R}^N) \rightsquigarrow \Gamma(S) = \Gamma(M, S) \qquad \begin{array}{c} S \\ \downarrow \\ M \end{array} \text{ spinor bundle}$$

spinor fields on M $\qquad\qquad$ $w_2(M) = 0$

$$\partial/\partial x_i \rightsquigarrow \nabla_{\partial_i}$$

$$e_i \cdot \rightsquigarrow X \cdot \in \Gamma(\mathrm{End}(S)) \text{ for } X \in \mathfrak{X}(M).$$

Dirac operator on (M, g)

$$\boxed{D = \sum e_i \cdot \nabla_{e_i} : \Gamma(S) \to \Gamma(S)}$$

1-st order elliptic selfadj operator

M = even $\Rightarrow$ $S = S^+ \oplus S^-$. $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$

Harmonic spinor (Dirac field) $\quad M$ cpt

④

$$H(D) := \{ \varphi \in \Gamma(S) \mid D\varphi = 0 \} \quad (= H(D^2))$$

(describing fermions with spin 1/2 in physics)

- M^{2n} : $\quad H(D) = H(D^+) \oplus H(D^-)$

$$\boxed{\begin{aligned} \operatorname{ind} D &= \dim H(D^+) - \dim H(D^-) \\ &= \int_M \hat{A}(M) = \hat{A}(M) \qquad \text{Atiyah-Singer} \end{aligned}}$$

<u>Note</u> $\dim H(D)$ is not top inv.

- Lichnerowicz (B.W formula)

$$D^2 = \nabla^* \nabla + \frac{1}{4} \mathrm{Scal}$$

$$\boxed{\begin{gathered} \text{vanishing} \\ \mathrm{Scal} > 0 \Rightarrow H(D) = \{0\} \end{gathered}}$$

Spin geometry (geometry by spinors and Dirac op) 5

Parallel spinor $\quad \nabla\varphi = 0$

Bergeの分類

$\Rightarrow \quad Ric = 0$

$Hol(M) =$	$SU(m)$	$Sp(k)$,	G_2,	$Spin(7)$
	C-Y	HK	7-dim	8-dim
	2m-dim	4k-dim		

Killing spinor $\quad \nabla_X \varphi = c\, X\cdot\varphi \quad \forall X \in \mathfrak{X}(M) \quad (c = \pm\tfrac{1}{2})$

$\Rightarrow \quad Ric = 4(n-1)c^2 g \qquad$ (Einstein mfd)

Geometric Str = Ein-Sasaki, 3-Sasaki,
Nearly kähler, Nearly parallel G_2

Cone

harmonic spinor with spin $k/2$

		h.w of $Spin(n)$
Spin $\frac{1}{2}$:	Dirac field	$(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$
Spin 1 :	harmonic 1-form	$(1, 0, \cdots, 0)$
Spin $\frac{3}{2}$:	Rarita-Schwinger field (1941)	$(\frac{3}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$
Spin 2 :	linearlized gravity	$(2, 0, \cdots, 0)$

$(\overbrace{1, \cdots, 1}^{p}, 0, \cdots 0)$: harmonic p-form ($\rightsquigarrow$ killing p-form)

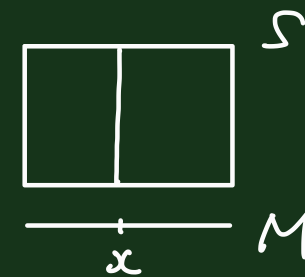
$(p, 0, \cdots, 0)$: harmonic (tr $= 0$) sym p-tensor ($\rightsquigarrow$ killing tensor)

$$\left.\begin{array}{l} (\frac{k}{2}, \frac{1}{2}, \cdots, \frac{1}{2}) \\ (\frac{3}{2}, \cdots, \frac{3}{2}, \frac{1}{2}, \cdots, \frac{1}{2}) \end{array}\right\} \text{higher spin field}$$

Def of R-S operator, R-S fields

(M, g) : Riem spin mfd

$S \to M$: Spinor bundle on M



$S \otimes TM \cong S \otimes T^*M$ $\quad (TM = S_1)$

$\cong S \oplus S_{3/2}$ $\quad$ irr decomp

$$\begin{tikzcd}
 & & \Gamma(S) \\
\Gamma(S) \arrow[r, "\nabla"] \arrow[urr, bend left, "D : \text{Dirac op}"] \arrow[drr, bend right, "P : \text{Penrose (twistor) op}"'] & \Gamma(S \otimes T^*M) \arrow[ur, "pr"] \arrow[dr, "pr"'] & \\
 & & \Gamma(S_{3/2})
\end{tikzcd}$$

$pr(\varphi \otimes e) = \frac{1}{\sqrt{n}} e \cdot \varphi$

∇ "=" $D + P$

$$S_{3/2}\otimes T^*M \cong S_{3/2}\oplus S\oplus S_{5/2}\oplus S_{3/2,3/2}.$$

$$\nabla = Q + P^* + P_{5/2} + P_{3/2,3/2}$$

Q : Rarita-Schwinger op

$$\begin{tikzcd}
 & & \Gamma(S_{3/2}) \\
\Gamma(S_{3/2}) \arrow[r, "\nabla"] \arrow[urr, bend left, "Q"] \arrow[drr, bend right, "P^*"'] & \Gamma(S_{3/2}\otimes T^*M) \arrow[ur] \arrow[dr] & \\
 & & \Gamma(S) \arrow[ll, bend left=10, "P"']
\end{tikzcd}$$

P^* : adj of P

Q is 1-st order self ad elliptic, but $\sigma(Q^2)\neq\sigma(\Delta)$

P is overdetermined elliptic, i.e. P^*P is elliptic

$\hookrightarrow \Gamma(S_{3/2}) \cong \ker P^* \oplus \mathrm{Im}(P)$

R-S op via twisted Dirac op

$$D_{TM} : \Gamma(\underline{S \otimes TM}) \to \Gamma(S \otimes TM) \quad \text{by } \sum_{i=1}^{n} (e_i \cdot \otimes I) \nabla_{e_i}$$

$$S \otimes TM = S \oplus S_{3/2}$$

$$\therefore \quad D_{TM} = \begin{pmatrix} \frac{2-n}{n} D & 2P^* \\ 2/n\, P & Q \end{pmatrix} \quad \star$$

twisted Lichnerowicz

$$D_{TM}^2 = \Delta + Scal/8 - I \otimes Ric = \begin{pmatrix} \Delta + \frac{n-8}{8n} Scal & 2\left(Ric - \frac{1}{n} g\right)^* \\ 2\left(Ric - \frac{1}{n} g\right) & \Delta + Scal/8 - Ric_{3/2} \end{pmatrix}$$

$$\star^2 \, \| \qquad \qquad \|$$

$$\begin{pmatrix} \frac{(n-2)^2}{n^2} D^2 + \frac{4}{n} P^* P & 2\left(P^* Q - \frac{n-2}{n} D P^*\right) \\ \frac{2}{n}\left(QP - \frac{n-2}{n} PD\right) & Q^2 + \frac{4}{n^2} PP^* \end{pmatrix}$$

$$D_{TM} D_{TM}^2 = D_{TM}^2 D_{TM}$$

Prop (M,g) cpt Einstein spin mfd

$$\begin{cases} Q^2 + \frac{4}{n} PP^* = \Delta + \frac{n-8}{8n} \mathrm{Scal} \\ QP = \frac{n-2}{n} PD, \quad P^*Q = \frac{n-2}{n} DP^* \\ \Delta P = P\Delta, \quad P^*\Delta = \Delta P^*, \quad Q\Delta = \Delta Q \end{cases}$$

$$\Gamma(S) = \ker P^* \oplus \mathrm{Im}\, P \ \circlearrowleft Q, \Delta$$

In general,
$\Delta = \Delta_{3/2} \not\geq 0$
(similar to $\Delta_2 \not\geq 0$)
$\Downarrow$
vanishing thm is not easy.

$$Q^2 = \begin{cases} \Delta + \frac{n-8}{8n} \mathrm{scal} & \text{on } \ker P^* \\ \left(\frac{n-2}{n}\right)^2 (\Delta + \mathrm{Scal}/8) & \text{on } \mathrm{Im}\, P \end{cases}$$

where Δ : Standard Lap (on G/K, Casimir)

Ravita - Schwinger field

φ : R-S field $\Leftrightarrow$ $\varphi \in \Gamma(S_{3/2})$, $D_{TM}\varphi = 0$

$\Leftrightarrow$ $\varphi \in \Gamma(S_{3/2})$, $Q\varphi = 0$, $P^*\varphi = 0$

$RS(M) := \dim\{\varphi \mid \varphi : \text{R-S fields}\}$

- $\dim M$ = even. cpt Einstein spin

$$RS(M) \geqq |RS^+(M) - RS^-(M)| = \left| \int_M \hat{A}(M)\, ch(TM) \right|$$

- M cpt Einstein spin $Scal \geqq 0$

$$\Rightarrow RS(M) = \dim \ker Q$$

Some cpx mfds with R-S fields

[H. Semmelmann, 2019]

$M := \underline{X_m(d)} \subset \mathbb{C}P^{m+1}$ $\quad$ <u>one homog poly deg = d</u>

Tian (1987) $\quad X_6(4),\ X_6(6)$ $(\dim M = 12)$

has <u>positive Ein-kähler str.</u>

Hirzebruch $\quad \hat{A}(M) = 0, \quad \sigma(M) \neq 0$

$$\int \hat{A}(M)\,ch(TM) = 5\hat{A}(M) + \sigma(M)/8 \neq 0$$

$\therefore$ $\boxed{RS(M)\ (= \dim \ker Q) > 0}$

$$Q^2 = \Delta_{3/2} + \underbrace{\frac{n-8}{8n}\,scal}_{>0} \quad \text{on } \ker P^*$$

$\Delta_{3/2} \neq 0\ !!$

positive Ein $\Rightarrow H(D) = \{0\}$

[Bär - Mazzeo, CMP 2021]

In the same method,

$$\boxed{\begin{array}{l}\text{For } n \geqq 4,\ C>0, \\ \exists M^n : \text{cpt Riem spin mfd with } RS > C\end{array}}$$

$\because)$ $M = X_m(d) \subset \mathbb{C}P^{m+1}$

- $d = \text{even} \Rightarrow \text{Spin}$
- $d = m+2 \Rightarrow c_1(M) < 0$ By Calabi-Yau $\exists$ negative Ein-kähler metric on M
- $d \gg 0 \Rightarrow \left|\int_M \hat{A}(M)\, ch(TM)\right| > C$

$M = T^k \times X_m(d)$ $\dim = 2m+k$ //

RS-fields on cpt sym sp

$\Delta_{3/2}$ = Casimir op on G/K $\therefore$ $\Delta \geqq 0$ on G/K

<u>vanishing thm</u> G/K cpt type irr sym sp dim > 8

($\therefore$ Einstein)

$\Rightarrow$ $RS(G/K) = 0$

$\because$) $Q^2 = \underbrace{\Delta_{3/2}}_{\geqq 0} + \underbrace{\frac{n-8}{8n}\mathrm{Scal}}_{>0}$

Thm [H.S, 2019]

G/K irr cpt type symm space with RS > 0

$\Rightarrow$ $G/K = Gr_2(\mathbb{C}^4),\ \mathbb{H}P^2,\ G_2/SO(4)$ ($\rightsquigarrow$ q-kähler)

$SU(3)$ (PSU(3)-str)

Spectra of Q on cpt type irr Sym space G/K ⑮

$$\Gamma(S_{3/2}) = \underbrace{\mathrm{Im}\,P}_{\cong\ \Gamma(S)\ominus \underline{\ker P} = 0 \text{ on } M\neq S^n} \oplus \ker P^* \qquad \Delta_{3/2} = \text{Casimir}$$

$$Q^2 = \begin{cases} \Delta_{3/2} + \dfrac{n-8}{8n}\,\mathrm{scal} & \text{on } \ker P^* \\ \left(\dfrac{n-2}{n}\right)^2 \left(\Delta_{3/2} + \mathrm{scal}/8\right) & \text{on } \mathrm{Im}\,P \end{cases}$$

By usual method, we calculate $\mathrm{Spect}(Q^2)$

[H- Tomihisa. 2021]

We have spectra of Q on S^n, $\mathbb{C}P^{2m+1}$, $\mathbb{H}P^k$

R-S fields on q.k mfd

quaternionic kähler mfd (M^{4k}, g)

$\mathrm{Hol}(M) = Sp(1)\,Sp(k)$

$TM \cong H \otimes E$

$\rightsquigarrow S_{3/2} \cong \bigoplus_{d,a,b} S^d(H) \otimes \Lambda_0^{a,b}(E)$

irr vector bdle

[H 2006, Sem-Weingart 2002]

$$\Delta \geq \frac{\mathrm{Scal}}{8k(k+2)}(d+a-b)(d-a-b+2m+2)$$

on $S^d(H) \otimes \Lambda_0^{a,b}(E)$

$$\therefore Q^2 = \Delta + \frac{n-8}{8n}\mathrm{Scal} > 0 \quad (n \neq 8)$$

Thm [H-S] M positive q-kähler mfd with RS>0

$\Rightarrow M = Gr_2(\mathbb{C}^4),\ \mathbb{H}P^2,\ G_2/SO(4)$

R.S fields on Ricci flat mfd

M : irr Riem, (simply conn), $Ric = 0$ ($\therefore$ $scal = 0$)

$\Rightarrow$ $Hol(M) = SU(n), Sp(n), G_2, Spin(7)$
7dim 8-dim

<u>Ex</u> (M^7, g) with $Hol(M) = G_2$

$Q^2 = \Delta$ and $RS = \dim\ker\Delta$

bdle decomp w.r.t G_2

φ associative 3-form

$\Lambda^2 \cong \Lambda^1 \oplus \Lambda^2_{14}$, $\Lambda^3 \cong \underline{\mathbb{C}} \oplus \Lambda^1 \oplus \Lambda^3_{27} \oplus \Lambda^3_{48}$

$S \cong \mathbb{C} \oplus \Lambda^1$. $S_{3/2} \cong \Lambda^1 \oplus \Lambda^3_{27} \oplus \Lambda^2_{14}$

$\Delta_{3/2} = d\delta + \delta d$ on $\Lambda^\bullet$ より, $\underline{RS = b_2(M) + b_3(M) - 1}$

$\rightsquigarrow$ $\exists$ G_2 mfd with $RS > 0$

Ex (M^{2n}, g) Calabi-Yau mfd $\mathrm{Hol}(M) = SU(n)$

$$RS = -2 + 2 \times \sum_{1 \leq p \leq n-1} h^{1,p}$$

$h^{p,q}$: Hodge number

$(= \dim \ker \Delta \text{ on } \Lambda^{p,q})$

- M K3-surface $(n=2)$

$$RS = -2 + 2h^{1,1} = 38$$

- M cpx 3-dim Calabi-Yau

$$RS = 2(h^{1,1} + h^{1,2}) - 2 = 2b_2(M) + b_3(M) - 4$$

Thm [H-S]

$\exists$ M cpt mfd with $\begin{cases} \mathrm{Hol}(M) = SU(m), Sp(k), G_2 \text{ or } \mathrm{spin}(7) \\ RS > 0 \end{cases}$

RS-fields on Nearly kähler mfd

(M, g, J) N-K 6-dim mfd (Scal = 30)

- almost Hermitian
- $(\nabla_X J)(X) = 0 \quad \forall X$

$\Rightarrow \exists\, K$: killing spinor

- $\bar{\nabla}_X Y := \nabla_X Y - J(\nabla_X Y) Y \times \frac{1}{2}$ Herm conn

$(\bar{\nabla} g = 0,\ \bar{\nabla} J = 0,\ \bar{T}(X, Y) = -J(\nabla_X J))$

$\rightsquigarrow S \cong \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^6$ w.r.t. $\bar{\nabla}$, $S \otimes TM \cong \cdots$

Compare D_{TM}^2 and $\overline{D_{TM}}^2$ + long calculation on N-K mfd

$$\boxed{\begin{array}{l}\underline{\text{Thm}}\ \text{[Ohno - Tomihisa 2022]} \qquad \text{3-rd Betti} \\ (M^6, g, J)\ \text{cpt N-K mfd} \Rightarrow RS = b_3(M)\end{array}}$$

Ex homog N-K mfd

$$S^6 = G_2/SU(3), \quad S^3 \times S^3 = \frac{SU(2)\times SU(2)\times SU(2)}{\Delta SU(2)}$$

$$\mathbb{C}P^3 = \frac{Sp(2)}{U(1)\times Sp(1)}, \quad F(1,2) = SU(3)/T^2$$

+ Foscolo-Haskins's inhomog N-K str on S^6, $S^3\times S^3$

By $b_3(S^3\times S^3) = 2$

$$\therefore \quad RS(S^3\times S^3, \text{N-K}) = 2$$

But $S^3\times S^3$ with standard metric

$$RS(S^3\times S^3, \text{std}) = 0$$

$H(D)=0$

$H(\Delta \text{ on } \Lambda^p)$

も一致

RSで初めて区別できる.

Other problem, Future problem

- Deformation of Killing spinors [Wang, ⋯]

  ($\rightsquigarrow$ infin deform of Einstein metric)

- RS fields on Nearly Parallel G_2 mfd (M^7, g) [Ohno]

- Boundary value problem [Bär]

- (RS fields on Ein-Sasaki ??)

- Clifford analysis (on $\mathbb{R}^n$)

  (Poly sol on $\mathbb{R}^n$, Cauchy kernel, ⋯)

  [Eelbode, Soucék, ⋯]

- From Physics, - - - - -

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