スピン幾何学

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§1. overview

Laplace op $\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$ on $\mathbb{R}^n$

$\left(\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right.$ on $\left. \mathbb{R}^{1,3} \right)$

$\downarrow$ 物理からの要請
(電子など spin 1/2 粒子の記述)

Find 1-st order diff op D s.t. $D^2 = \Delta$

idea

Δ on $C^\infty(\mathbb{R}^n)$ $\rightsquigarrow$ D on $C^\infty(\mathbb{R}^n, W)$

vector valued fn
(spinor)

$$D := \sum_{i=1}^{n} \gamma_i \frac{\partial}{\partial x_i} \qquad \gamma_i \text{ 行列} \in \mathrm{End}(W)$$

$$D^2 = \sum_{i,j} \gamma_i \gamma_j \frac{\partial^2}{\partial x_i \partial x_j} = \Delta$$

$\therefore$ $\{\gamma_i\}$ は $\quad 1 \leq i, j \leq n$

$$\gamma_i \gamma_j + \gamma_j \gamma_i = -2\delta_{ij} \quad \text{(Clifford relation)}$$

Dirac field $\varphi \in C^\infty(\mathbb{R}^n, W)$ s.t. $D\varphi = 0$

Ex on $\mathbb{R}^3$ $\quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Pauli

$\gamma_i = \sqrt{-1}\sigma_i$, $W = \mathbb{C}^2$ $\qquad D = \sum_{i=1}^{n} \sqrt{-1}\sigma_i \frac{\partial}{\partial x_i}$

数学として見ると

$\{\gamma_i \gamma_j\} \rightsquigarrow$ Clifford algebra $Cl(V)$

$$\Lambda^* V \qquad v \wedge w + w \wedge v = 0$$

$$\Big\Downarrow$$

$$Cl(V) \qquad v \cdot w + w \cdot v = -2(v.w)$$

$W \rightsquigarrow$ spinor space

$\mathrm{Spin}(n) = \widetilde{SO(n)}$ の表現空間

Ex $\quad \underset{\substack{\| \\ S^3}}{\mathrm{Spin}(3) = SU(2)} \to \underset{\substack{\| \\ \mathbb{RP}^3}}{SO(3)}$ double cover

$C^\infty(\mathbb{R}^n, W) \rightsquigarrow \Gamma(M, S)$ $\quad \begin{array}{c} S \\ \downarrow \\ M \end{array}$ Spinor bundle

Sp of spinor fields.

$(M.g)$ には, spin str が必要

$(w_2(M) = 0)$

$D = \sum \gamma_i \frac{\partial}{\partial x_i} \rightsquigarrow D = \sum e_i \cdot \underbrace{\nabla_{e_i}}$ $\quad \{e_i\}$ o.n. frame of TM

Levi-Civita からの conn

"spin conn"

$D^2 = \Delta \rightsquigarrow D^2 = \underbrace{\nabla^* \nabla}_{\text{conn Laplacian}} + \frac{1}{4} \underbrace{\mathrm{Scal}}_{\text{scalar curvature}}$

$\mathrm{Scal} > 0 \Rightarrow H(D) = \{\varphi \in \Gamma(S) \mid D\varphi = 0\} = \{0\}$

$M^{2n} \Rightarrow S = S^+ \oplus S^-, \quad D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$

$\mathrm{ind}\, D := \dim_{\mathbb{C}} H(D^+) - \dim_{\mathbb{C}} H(D^-)$ (analytic index)

Atiya-Singer index thm

$$\mathrm{ind}(D) = \int_M \hat{A}(TM) = \hat{A}(M) \qquad \hat{A}\text{-genus of } M$$

M の diff topo invariant なの

D_E twisted Dirac $\Gamma(S\otimes E) \to \Gamma(S\otimes E)$

$$\mathrm{indx}\,(D_E) = \int_M \hat{A}(TM)\,\mathrm{ch}(E)$$

Ex $E = S^*$

- $\chi(M) = \int_M e(TM)$ ガウス-ボンネ-チャーン
- $\sigma(M) = \int_M L_k(TM)$ signature thm

応用 M 4-dim spin mfd (cpt. $\partial M = \phi$) のとき

S : fiber に四元数構造が入る

$\dim_{\mathbb{C}} H(D^{\pm}) \in 2\mathbb{Z}$ $\therefore$ $\hat{A}(M) \in 2\mathbb{Z}$

一方 $\hat{A}(M) = -\frac{1}{8}\sigma(M)$

$\therefore$ $\sigma(M)$ は 16の倍数 (ロホリンの定理)

応用 $\mathrm{Scal} > 0 \Rightarrow \hat{A}(M) = 0$

M が $w_2(M) = 0$, $\hat{A}(M) \neq 0$ なら, 正のスカラー曲率の Riem metric g は入らない.

Eigenvalue estimate and limiting mfd

Δ on (M, g) cpt oriented Riem

Lichnerowicz - Obata

$$\mathrm{Ric} \geqq (n-1) r > 0 \quad (r \in \mathbb{R}) \Rightarrow \quad \underbrace{\lambda(\Delta)}_{\text{eign of }\Delta} \geqq nr$$

$$\text{"="} \Leftrightarrow (M, g) \underset{\text{isome}}{\cong} (S^n(r), g_0)$$

$$D^2 = \nabla^* \nabla + \frac{1}{4} \mathrm{Scal} \qquad (M, g) \text{ cpt spin mfd}$$

$$\rightsquigarrow \qquad \lambda(D^2) \geqq \frac{1}{4} \min_{x \in M} \mathrm{Scal}$$

Bochner-Weitzenböck formula

$$\frac{1}{2} \overbrace{P^* P}^{\geqq 0} - \frac{n-1}{2n} D^2 = -\frac{1}{8} \mathrm{Scal}$$

Penrose (twistor) op

Friedrich's estimate

$$\lambda(D^2) \geqq \frac{n}{4(n-1)} \min_{x \in M} (\mathrm{Scal}(x))$$

$$\text{"="} \Leftrightarrow \qquad \nabla_X \psi = c X \cdot \psi \quad (\forall X \in \mathfrak{X}(M))$$

Killing spinor

$\Rightarrow$ (M, g) cpt Einstein spin mfd

with $\mathrm{Ric} = 4(n-1) c^2 g \qquad (c \in \mathbb{R})$

Special spinors

parallel spinor φ $\quad \nabla \varphi = 0$

$\Rightarrow$ $\mathrm{Hol}(M) = SU(m),\ Sp(k),\ G_2,\ Spin(7)$

Calabi-Yau, hyper kähler, 7-dim, 8-dim

[Wang 1989]

real killing spinor φ $\quad \nabla_X \varphi = c X \cdot \varphi$

$(c \in \mathbb{R})$

$\Rightarrow$ $C(M)$ cone of M

$\mathrm{Hol}(C(M))$ is above.

$\rightsquigarrow$ (M, g): Ein-Sasaki, 3-Sasaki, Nearly Kähler (6-dim)

Nearly Parallel G_2 (7-dim)

[Bär 1993]

imaginary killing spinor $\quad \nabla_X \varphi = \sqrt{-1}\, c X \cdot \varphi$

$(c \in \mathbb{R})$

(M, g) is warped product

$$(F^{n-1} \times \mathbb{R},\ e^{-4ct} g_F + dt^2)$$

[Baum 1989]

s.t. F has parallel spinor.

Math related to spin geometry $\rightsquigarrow$ spin $1/2$

Seiberg-Witten, twistor theory, special geometry

index thm, non-commutative geometry, Clifford analysis

…, and math physics.

時間があれば spin $3/2$ version

§2 Clifford algebra

V : n-dim vector space on $\mathbb{R}$ or $\mathbb{C}$

basis $\{v_i\}$

tensor algebra

$T^p(V) = V \otimes \cdots \otimes V \quad (p = 0.\ 1.\ 2 \cdots, \quad T^0(V) = \mathbb{R})$

basis $\{ v_{i_1} \otimes v_{i_2} \cdots \otimes v_{i_p} \mid i_1, \cdots, i_p = 1, \cdots n \}$

$T^*(V) = \bigoplus_{p=0}^{\infty} T^p(V)$ tensor algebra

by $T^p(V) \times T^q(V) \ni (\varphi.\ \psi) \mapsto \varphi \otimes \psi \in T^{p+q}(V)$

Note $T^*(V)$ is associative alg with 1

generated by $\{v_1, \cdots, v_n\}$

exterior algebra

$S = \{ v \otimes v \mid v \in V \}$

$I(S)$: two-sided ideal generated by S

$\Lambda^*(V) := T^*(V) / I(S) \xleftarrow[\text{proj}]{\pi} T^*(V)$

$\Lambda^p(V) := \pi(\otimes^p V)$

wedge product $\alpha \wedge \beta := \pi(a \otimes b)$ for $\alpha = \pi(a),\ \beta = \pi(b)$

ex $v \wedge v = 0, \quad (u+v) \wedge (u+v) = 0 \quad {}^\forall u.\ v \in V$

$u \wedge v + v \wedge u = 0 \qquad {}^{\forall} u, v \in V$

<u>Ex</u> $\quad \underline{u_1 \wedge \cdots \wedge u_p} \wedge \underline{v_1 \wedge \cdots \wedge v_q}$

$= (-1)^p v_1 \wedge u_1 \wedge \cdots \wedge u_p \wedge v_2 \wedge \cdots \wedge v_q$

$= \cdots = (-1)^{pq} \underline{v_1 \wedge \cdots \wedge v_q} \wedge \underline{u_1 \wedge \cdots \wedge u_p}$

$\therefore \quad \alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \quad (\alpha \in \Lambda^p(V),\ \beta \in \Lambda^q(V))$

もっと具体的に

$\Lambda^* V$: $v_1, \cdots, v_n, 1$ generators

$v_i \wedge v_j + v_j \wedge v_i = 0$ relations

とする $\mathbb{R}$上 algebra

$\rightsquigarrow$ basis of $\Lambda^p(V)$

$v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_p} \quad 1 \leqq i_1 < i_2 < \cdots < i_p \leqq n$

$\dim \Lambda^p(V) = \binom{n}{p}$

とくに、 $\dim \Lambda^n(V) = 1$ で

$v_1 \wedge \cdots \wedge v_n$ が basis である //

V に内積 or 計量を入れたら

<u>Hodge star operator</u>

$\langle\ ,\ \rangle$ (p.d.) inner product on V

$\longrightarrow \langle\ ,\ \rangle$ on $T^p(V)$, $\Lambda^p(V)$

Def $\alpha = u_1 \wedge \cdots \wedge u_p,\ \beta = w_1 \wedge \cdots \wedge w_p$

$\langle \alpha, \beta \rangle := \det(\langle u_i, w_j \rangle)_{i,j}$ on $\Lambda^p(V)$

あとは linear に拡張

ex $\{e_i\}$: o.n.b for V

$\{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid 1 \leqq i_1 < \cdots < i_p \leqq n\}$ o.n.b for $\Lambda^p(V)$

Def $\Lambda^n V \setminus \{0\}$ 2-conn comp

1つの conn comp を定める = V に向きを入れる

$\downarrow$

$\underset{\neq 0}{v_1 \wedge \cdots \wedge v_n}$ を正の向きという.

$(v_1, \cdots, v_n)$ を positive basis

Def $V, \langle\ ,\ \rangle$ + orientation

$(e_1, \cdots, e_n)$ positive orthnormal basis

このとき $\mathrm{vol} := e_1 \wedge \cdots \wedge e_n$ volume element という

Exc vol は. p.o.n.b のとり方によらない.

Def Hodge star operator

$* : \Lambda^p(V) \longrightarrow \Lambda^{n-p}(V)$ linear isom を

def by $\phi \wedge * \psi = \langle \phi, \psi \rangle \mathrm{vol}$ $(\phi, \psi \in \Lambda^p(V))$

もっと具体的に

$(e_1, \cdots, e_n)$ p.o.n.b
n.o.n.b

$*1 = \underline{\pm}\, e_1\wedge\cdots\wedge e_n$, $*(e_1\wedge\cdots\wedge e_n) = \underline{\pm}\, 1$

とする

$*(e_1\wedge\cdots\wedge e_p) = \underline{\pm}\, e_{p+1}\wedge\cdots\wedge e_n$

<u>Ex</u> $(e_1, \cdots, e_5)$ p.o.n.b

$*(e_2\wedge e_4\wedge e_5) = \underset{?}{\pm}\, e_1\wedge e_3 \quad = -e_1\wedge e_3$

$(e_2, e_4, e_5, e_1, e_3)$ は negative

<u>Exc</u> (1) $*^2 = (-1)^{p(n-p)}$ on $\Lambda^p(V)$

(2) $\langle \phi, \psi\rangle = *(\phi\wedge *\psi) = *(\psi\wedge *\phi)$

$= \langle *\phi, *\psi\rangle$

<u>Exc</u> $V \ni v \mapsto \langle v, \cdot\rangle \in V^*$ linear isom

$(\mathbb{R}^3, g_E)$ $T_x(\mathbb{R}^3) \cong T_x^*(\mathbb{R}^3)$

$\frac{\partial}{\partial x_i} \longleftrightarrow dx_i$

この同一視のもとで

$*d : \Omega^1(\mathbb{R}^3) \xrightarrow{d} \Omega^2(\mathbb{R}^3) \xrightarrow{*} \Omega^1(\mathbb{R}^3)$

$*d* : \Omega^1(\mathbb{R}^3) \xrightarrow{*} \Omega^2(\mathbb{R}^3) \xrightarrow{d} \Omega^3(\mathbb{R}^3) \xrightarrow{*} C^\infty(\mathbb{R}^3)$

を考える $*d = \mathrm{rot}$, $*d* = \mathrm{div}$ を示せ

Clifford algebra

$V, \langle\, , \rangle$

$S' := \{ v \otimes v + \langle v, v \rangle 1 \mid v \in V \} \subset T^*(V)$

$J(S')$: ideal by S'

Def $\quad Cl(V) := T^*(V) / J(S') \xleftarrow{\pi} T^*(V)$

$\alpha \cdot \beta \ (= \alpha\beta) := \pi(a \otimes b) \quad$ Clifford product

Prop $\quad$ product is bilinear, associative

$\because)$ $\otimes$ is bilinear, associative $/\!/$

Ex $\quad u, v \in V \qquad u \cdot u = -\|u\|^2$

$$
\begin{array}{ccc}
(u+v)\cdot(u+v) & = & -\|u+v\|^2 \\
\| & & \| \\
u^2 + uv + vu + v^2 & & \|u\|^2 + 2\langle u, v\rangle + \|v\|^2
\end{array}
$$

$\therefore \quad uv + vu = -2\langle u, v\rangle \quad$ Clifford relation

とくに, $\{e_i\}_{i=1}^n$ o.n.b

$$e_i e_j + e_j e_i = -2\delta_{ij}$$

Lem $\quad \pi(T^0(V)) = \pi(\mathbb{R}) = \mathbb{R} \subset Cl(V)$

$\pi(T^1(V)) = \pi(V) = V \subset Cl(V)$

$\because)$ $\pi|_{\mathbb{R}}, \ \pi|_V$ は inj $\quad$ (proof Page 186)

Note $Cl(V)$ is not graded alg.

$\because$) $u \in V$, $u \cdot u \in \pi(T^2(V))$
$\qquad\qquad\qquad \| $
$\qquad -2\|u\|^2 \in \pi(T^0(V))$ //

あとで見るが $\mathbb{Z}_2$-graded str は入る.

もっと具体的に

$Cl(V)$: generators $e_1, \cdots, e_n, 1$

relations $e_i e_j + e_j e_i = -2\delta_{ij}$

とする $\mathbb{R}$上 algebra

Ex $1 - 3e_1e_2 + \sqrt{2}\, e_3e_4e_7 \in Cl(V)$

Ex $e_4e_1e_2e_1e_3e_2 = -e_1e_2e_1e_3e_2e_4$

$= -(-1)(-1)\, e_1^2 e_2^2 e_3e_4 = -(-1)^4 e_3e_4$

$= -e_3e_4$

Ex $(1 - e_2 + e_1e_3)(-e_1 + 2e_2)$

$= -e_1 + e_2e_1 - e_1e_3e_1 + 2e_2 - 2e_2^2 + 2e_1e_3e_2$

$= 2 - e_1 + 2e_2 - e_3 - e_1e_2 - 2e_1e_2e_3$

basis of $Cl(V)$ は $\qquad$ as vector sp $Cl(V) \cong \Lambda^*(V)$

$\{e_{i_1}e_{i_2}\cdots e_{i_p} \mid 1 \leqq i_1 < \cdots < i_p \leqq n,\ p = 1, \cdots n\}$ のようだ

prop (universality)

A : associative algebra with 1 over $\mathbb{R}$

$f : V \to A$ linear s.t. $f(v)f(v) = -\langle v, v\rangle 1_A$

$\Rightarrow$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, hook, "i"'] & A \\ Cl(V) \arrow[ur, "\exists 1 F"'] & \end{tikzcd}$$ alg homo

proof tensor alg $T^*(V)$ の universality より

$\exists 1 \; F' : T^*(V) \to A$ s.t. $F'|_V = f$

$\because)$ $F'(v_1 \otimes \cdots \otimes v_p) = f(v_1)\cdots f(v_p)$ とすればよい.

さて, $f(v)f(v) = -\langle v.v\rangle 1_A$ より

$F'(J(S')) = 0$

$$\therefore \quad \begin{tikzcd} T^*(V) \arrow[r, "F'"] \arrow[d] & A \\ Cl(V) \arrow[ur, "F"'] & \end{tikzcd} \qquad F|_V = f$$

$Cl(V)$ は V で生成されるので, F は unique //

Ex $\dim V = 1$ e_1 o.n.b

$f : V \ni e_1 \to \sqrt{-1} \in \mathbb{C}$ linear (on $\mathbb{R}$), $f(e_1)^2 = -1$

$\therefore$ $F : Cl(V) \to \mathbb{C}$ alg homo

$\mathbb{C}$ の $\mathbb{R}$上alg としての generator $\sqrt{-1}, 1$ $\therefore$ $\boxed{Cl(V) \cong \mathbb{C}}$

Ex $\dim V = 2$ o.n.b e_1, e_2

Def (quaternion)

$$\mathbb{H} = \{ a1 + bi + cj + dk \mid a, b, c, d \in \mathbb{R} \} \cong \mathbb{R}^4$$

$$i^2 = j^2 = -1 \qquad ij = -ji = k$$

$$(\therefore k^2 = -1, \ jk = -kj = i, \ ki = -ik = j, \ ijk = -1)$$

$\mathbb{H}$ is non-comm associative alg on $\mathbb{R}$

Exc $p = a1 + bi + cj + dk \to \bar{p} = a1 - bi - cj - dk$

とする. quaternionic conjugate

(1) $\overline{pq} = \bar{q}\,\bar{p}$, (2) $p\bar{p} = \|p\|^2$

(3) $\|pq\| = \|p\|\,\|q\|$ (4) $p \neq 0 \Rightarrow p^{-1} = \dfrac{\bar{p}}{\|p\|^2}$ //

$f : V \to \mathbb{H}$ を $f(e_1) = i, \ f(e_2) = j$

$\rightsquigarrow \exists^1 F : Cl(V) \to \mathbb{H}$ alg isom

generator $e_1, e_2, 1$

$$\boxed{Cl(V) \cong \mathbb{H}}$$

Ex $\dim V = 3$ e_1, e_2, e_3

$$e_1 \mapsto (i, -i), \ e_2 \mapsto (j, -j), \ e_3 \mapsto (k, -k)$$

$$\boxed{Cl(V) \cong \mathbb{H} \oplus \mathbb{H}} \quad \text{alg isom}$$

$$(\ 1 \mapsto (1, 1) \ )$$

Def (1) alg A is $\mathbb{Z}_2$-graded とは.

$$A = \underset{\text{even}}{A^0} \oplus \underset{\text{odd}}{A^1}, \quad A^i A^j \subset A^{i+j \bmod 2}$$

(2) A, B $\mathbb{Z}_2$-graded alg

$A \hat{\otimes} B$ $\mathbb{Z}_2$-graded tensor product alg

(2-a) vector sp として $A \otimes B$

(2-b) $(a \otimes b)(a' \otimes b') = (-1)^{|a'||b|} aa' \otimes bb'$

$(\ |a| = \deg(a)\ )$

(2-c) $(A \hat{\otimes} B)^0 = (A^0 \otimes B^0) \oplus (A^1 \otimes B^1)$

$(A \otimes B)^1 = (A^0 \otimes B^1) \oplus (A^1 \otimes B^0)$

Ex $$Cl(V) = \underset{\text{even}}{Cl^0(V)} \oplus \underset{\text{odd}}{Cl^1(V)},$$

$Cl^0(V)$ is sub alg of $Cl(V)$ //

prop $V_1, \langle , \rangle$, V_2 $\langle , \rangle$

$Cl(V_1 \oplus V_2) \cong Cl(V_1) \hat{\otimes} Cl(V_2)$ $\mathbb{Z}_2$-grad alg homo

とくに $\dim Cl(V_1 \oplus V_2) = \dim Cl(V_1) \dim Cl(V_2)$

proof $f : V_1 \oplus V_2 \ni v_1 + v_2 \mapsto v_1 \otimes 1 + 1 \otimes v_2 \in Cl(V_1) \hat{\otimes} Cl(V_2)$

とすると, $f(v_1 + v_2)^2 = -\|v_1 + v_2\|^2 1 \otimes 1$ *

$\therefore F : Cl(V_1 \oplus V_2) \to Cl(V_1) \hat{\otimes} Cl(V_2)$ alg homo generator $e_1, \cdots, e_n \rightsquigarrow e_1\otimes 1, \cdots, e_n\otimes 1$

$e_1', \cdots, e_m' \qquad 1\otimes e_1', \cdots, 1\otimes e_m'$

* relations は同じ

$$\therefore \quad Cl(V_1 \oplus V_2) \cong Cl(V_1) \hat{\otimes} Cl(V_2) \,//$$

$V = L_1 \oplus \cdots \oplus L_n \quad L_i = \langle e_i \rangle_{\mathbb{R}}, \quad Cl(L_i) \cong \mathbb{C}$

$\therefore \quad Cl(V) \cong \mathbb{C} \hat{\otimes} \cdots \hat{\otimes} \mathbb{C} \qquad \therefore \quad \dim_{\mathbb{R}} Cl(V) = 2^n$

Ex $\quad \dim V = 3$

$$Cl(V) \cong \underset{\langle 1.\, e_1 \rangle}{Cl(L_1)} \hat{\otimes} \underset{\langle 1.\, e_2 \rangle}{Cl(L_2)} \hat{\otimes} \underset{\langle 1.\, e_3 \rangle}{Cl(L_3)}$$

$1\otimes 1\otimes 1, \quad e_1\otimes 1\otimes 1, \quad 1\otimes e_2\otimes 1, \quad 1\otimes 1\otimes e_3$

$e_1\otimes e_2\otimes 1, \quad e_1\otimes 1\otimes e_3, \quad 1\otimes e_2\otimes e_3, \quad e_1\otimes e_2\otimes e_3$

$\therefore$ basis of $Cl(V)$

$1, \; e_1, e_2, \; e_3, \; e_1e_2, \; e_1e_3, \; e_2e_3, \; e_1e_2e_3$

<u>Thm</u>

$$\Lambda^*(V) \ni e_{i_1} \wedge \cdots \wedge e_{i_p} \longmapsto e_{i_1} \cdots e_{i_p} \in Cl(V)$$

$$1 \leq i_1 < \cdots < i_p \leq n, \quad p = 0, \cdots, n$$

は1:1 linear isom (not alg isom)

∵ ⟸ $\quad \dim Cl(V) = 2^n$

Ex　$\dim V = 4$

$$Cl_4 \cong \mathbb{H}(2) = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \middle| \ p, q, r, s \in \mathbb{H} \right\}$$

$$e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$$

$$e_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\left\{ \begin{array}{lll} e_1 e_2 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} & e_1 e_3 = \begin{pmatrix} -j & 0 \\ 0 & -j \end{pmatrix} & e_1 e_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ e_2 e_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & e_2 e_4 = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix} & e_3 e_4 = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} \\ e_1 e_2 e_3 e_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \end{array} \right.$$

$\rightarrow$ basis of $Cl^0(V) \cong \mathbb{H} \oplus \mathbb{H} \cong Cl(\mathbb{R}^3)$

$$= \left\{ \begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix} \middle| \ p, s \in \mathbb{H} \right\}$$

$$Cl^1(V) = \left\{ \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \middle| \ q, r \in \mathbb{H} \right\}$$

V^n について は $\mathbb{R}^n$ と同一視して

$Cl(V) \cong Cl(\mathbb{R}^n) =: Cl_n$ と書く

<u>Lem</u> $\mathbb{R}$上 alg isom

$\mathbb{R}(m)\otimes\mathbb{R}(n)\cong\mathbb{R}(mn)$, $\mathbb{R}(n)\otimes k\cong k(n)$ $(k=\mathbb{R}, \mathbb{C}, \mathbb{H})$

$\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}\underset{(1)}{\cong}\mathbb{C}\oplus\mathbb{C}$, $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}\underset{(2)}{\cong}\mathbb{C}(2)$, $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{H}\underset{(3)}{\cong}\mathbb{R}(4)$

$\because)$ (1)

$$\mathbb{C}\oplus\mathbb{C}\ni\begin{array}{l}(1,0)\mapsto\frac{1}{2}(1\otimes1+\sqrt{-1}\otimes\sqrt{-1})\\(0,1)\mapsto\frac{1}{2}(1\otimes1-\sqrt{-1}\otimes\sqrt{-1})\end{array}\in\mathbb{C}\otimes\mathbb{C}$$

(3) $\mathbb{H}\otimes\mathbb{H}\ni p\otimes q\mapsto(\mathbb{H}\ni x\mapsto px\bar{q}\in\mathbb{H})\in\mathbb{R}(4)$
$(\mathbb{H}\cong\mathbb{R}^4)$

$$(p\otimes q)(p'\otimes q')(x)$$
$$=p(p'x\overline{q'})\bar{q}=(pp')x(\overline{qq'})$$
$$=(pp'\otimes qq')(x)$$

$\therefore$ $\mathbb{H}\otimes\mathbb{H}\to\mathbb{R}(4)$ alg homo

inj も easy dim $=16$ $\therefore$ alg isom //

<u>Exc</u> $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}\cong\mathbb{C}(2)$ を示せ

<u>Thm</u> $\mathbb{R}$上 alg isom

$Cl_1\cong\mathbb{C}$, $Cl_2\cong\mathbb{H}$, $Cl_3\cong\mathbb{H}\oplus\mathbb{H}$

$Cl_4\cong\mathbb{H}(2)$, $Cl_5\cong\mathbb{C}(4)$, $Cl_6\cong\mathbb{R}(8)$

$Cl_7\cong\mathbb{R}(8)\oplus\mathbb{R}(8)$, $Cl_8\cong\mathbb{R}(16)$

$n\geqq 9$ $Cl_{n+8}\cong Cl_n\otimes\mathbb{R}(16)$

outline

$\mathbb{R}^n \rightsquigarrow Cl_n$, $\quad \mathbb{R}^{0,n} \quad uv + vu = 2\langle u, v\rangle \rightsquigarrow Cl_{0,n}$

$$Cl_{0,n+2} \cong Cl_n \otimes Cl_{0,2}$$

by $f(e_i) = \begin{cases} e_i' \otimes e_1'' e_2'' & i = 1, \cdots n \\ 1 \otimes e_{i-n}'' & i = n+1, n+2 \end{cases}$

$\{e_i\}$ o.n.b of $\mathbb{R}^{0,n}$, $\{e_i'\}$ o.n.b of $\mathbb{R}^n$.

$\{e_i''\}$ o.n.b of $\mathbb{R}^{0,2}$

同様に $\quad Cl_{n+2,0} \cong Cl_{0,n} \otimes Cl_2$.

$Cl_2 \cong \mathbb{H}$, $\quad Cl_{0,2} \cong \mathbb{R}(2)$ by $e_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

∴

$\otimes \mathbb{H}$

$\otimes \mathbb{R}(2)$

$Cl_1 \cong \mathbb{C}$		$Cl_{0,1} \cong \mathbb{R} \oplus \mathbb{R}$
$Cl_2 \cong \mathbb{H}$		$Cl_{0,2} \cong \mathbb{R}(2)$
$Cl_3 \cong \underline{\mathbb{H} \oplus \mathbb{H}}$		$Cl_{0,3} \cong \mathbb{C}(2)$
$Cl_4 \cong \mathbb{H}(2)$		$Cl_{0,4} \cong \mathbb{H}(2)$
$Cl_5 \cong \mathbb{C}(4)$		$Cl_{0,5} \cong \mathbb{H}(2) \oplus \mathbb{H}(2)$
$Cl_6 \cong \mathbb{R}(8)$		$Cl_{0,6} \cong \mathbb{H}(4)$
$Cl_7 \cong \underline{\mathbb{R}(8) \oplus \mathbb{R}(8)}$		$Cl_{0,7} \cong \mathbb{C}(8)$
$Cl_8 \cong \mathbb{R}(16)$	$\cong$	$Cl_{0,8} \cong \mathbb{R}(16)$

$1 \mapsto (1,1)$
$e_1 \mapsto (1,-1)$

//

cpx Clifford algebra

Def $\mathbb{C}l_n := Cl_n \otimes \mathbb{C}$ cpx Clifford alg

($\mathbb{C}$上 asso alg with 1)

generators $e_1, \cdots, e_n$

relations $e_i e_j + e_j e_i = -2\delta_{ij}$ $i \cdot j$

$\mathbb{C}$上 alg より $e_i' = \sqrt{-1} e_i$ を generator

relation は $e_i' e_j' + e_j' e_i' = 2\delta_{ij}$

$\therefore$ $\mathbb{C}l_n \cong Cl_{0,n} \otimes \mathbb{C} \cong Cl_n \otimes \mathbb{C}$

Ex $\mathbb{C}l_1 \cong Cl_1 \otimes \mathbb{C} \cong \mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$ ← $\mathbb{C}$上 alg isom

$\cong Cl_{0,1} \otimes \mathbb{C} \cong (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{C}$

$\mathbb{C}l_2 \cong Cl_2 \otimes \mathbb{C} \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(2)$

$\cong Cl_{0,2} \otimes \mathbb{C} \cong \mathbb{R}(2) \otimes \mathbb{C}$

Thm $\mathbb{C}l_n \otimes \mathbb{C}l_2 \cong \mathbb{C}l_{n+2}$ (Bott periodicity)

$\mathbb{C}l_{2m} \cong \mathbb{C}(2) \otimes \cdots \otimes \mathbb{C}(2) \cong \mathbb{C}(2^m)$

$\mathbb{C}l_{2m+1} \cong \mathbb{C}(2^m) \otimes (\mathbb{C} \oplus \mathbb{C}) \cong \mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$

$\mathbb{C}$上 alg として split

<u>Def</u> $\{e_i\}$ o.n.b of $\mathbb{R}^n$

$\omega = e_1 \cdots e_n$ volume element in Cl_n

$\omega_{\mathbb{C}} = (\sqrt{-1})^{[\frac{n+1}{2}]} e_1 \cdots e_n$ cpx vol in $\mathbb{C}l_n$

<u>Exc</u> $\omega^2 = (-1)^{\frac{n(n-1)}{2}}$, $\omega_{\mathbb{C}}^2 = 1$

$\omega v = (-1)^{n-1} v \omega \quad (v \in V)$ //

Exc より $n = 2m+1$

$\omega_{\mathbb{C}} : \mathbb{C}l_n \ni \alpha \longmapsto \omega_{\mathbb{C}} \alpha \in \mathbb{C}l_n$,

$\mathbb{C}l_n = \mathbb{C}l_n^+ \oplus \mathbb{C}l_n^-$ $\pm$ eigen sp

$\varphi^\pm \in \mathbb{C}l_n^\pm$ とすると

$\varphi^+ \varphi^- = \omega_{\mathbb{C}} (\varphi^+ \varphi^-) = -\varphi^+ \varphi^-$ $\therefore$ $\varphi^+ \varphi^- = 0$

$\varphi_1^\pm \cdot \varphi_2^\pm \in \mathbb{C}l_n^\pm$

$\therefore$ $\mathbb{C}l_n \cong \mathbb{C}l_n^+ \oplus \mathbb{C}l_n^-$ as alg

$\cong \mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$

$\omega_{\mathbb{C}} v = v \omega_{\mathbb{C}}$ より $\omega_{\mathbb{C}}$ は $\mathbb{C}l_n$ の中心元

$\therefore$ $\omega_{\mathbb{C}} = (\alpha I, \beta I)$ $\alpha, \beta \in \mathbb{C}$

$\omega_{\mathbb{C}}^2 = 1$ より $\alpha^2 = \beta^2 = 1$

$\omega_{\mathbb{C}}$ の $\pm$ eige sp decomp $= \mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$

$\therefore$ $\boxed{\omega_{\mathbb{C}} = (I, -I)}$

$\alpha : V \ni v \mapsto -v \in V$

$\rightsquigarrow \quad \alpha : Cl_n \ni v_1 \cdots v_p \mapsto (-1)^p v_1 \cdots v_p \in Cl_n$ alg isom

$$\underset{\substack{\text{even}\\+1}}{Cl_n = Cl_n^0} \oplus \underset{\substack{\text{odd}\\-1}}{Cl_n^1} \quad \mathbb{Z}\text{-graded decomp}$$

α-eigen sp

さて $\alpha(\omega_{\mathbb{C}}) = -\omega_{\mathbb{C}}$ より ($n = 2m+1$)

$\alpha(Cl_n^{\pm}) = Cl_n^{\mp}$ & $Cl_n^{+} \overset{\alpha}{\cong} Cl_n^{-}$

$\therefore \quad Cl_{2m+1} = \mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$ と実現したら

$Cl_{2m+1}^0 = \{(A, A) \mid A \in \mathbb{C}(2^m)\}$

$Cl_{2m+1}^1 = \{(A, -A) \mid A \in \mathbb{C}(2^m)\}$

となる//

§ Spin group

$$O(n) = \{ A \in \mathbb{R}(n) \mid {}^tAA = A{}^tA = I \}$$
$$\cup$$
$$SO(n) = \{ A \in O(n) \mid \det A = 1 \}$$

Fact

$SO(n)$: conn cpt Lie group

identity conn comp of $O(n)$

$\pi_1(SO(n)) \cong \mathbb{Z}_2 \quad (n \geqq 3)$

Def the n-th spin group $Spin(n)$ is

a Lie group such that $(n \geqq 3)$

- connected, simply connected
- double covering group of $SO(n)$

$Ad : Spin(n) \to SO(n)$ surj Lie gr homo

$\ker Ad = \mathbb{Z}_2$

Ex $Sp(1) = \{ p \in \mathbb{H} \mid \bar{p}p = 1 \} \cong S^3$

$p, q \in Sp(1)$, $|pq| = |p||q| = 1$. $|p^{-1}| = \left|\frac{\bar{p}}{|p|}\right| = 1$

$\rightsquigarrow$ Lie group

$$Sp(1) \ni p = \alpha + \beta j \overset{\cong}{\longmapsto} \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2) \quad \text{Lie gr isom}$$
$(\alpha, \beta \in \mathbb{C})$

where $SU(2) = \{ A \in \mathbb{C}(2) \mid A^*A = AA^* = I,\ \det A = 1 \}$.

$$= \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \ \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\} \cong S^3 \ //$$

$Sp(1) \curvearrowright \operatorname{Im} \mathbb{H} = \{ q \in \mathbb{H} \mid \bar{q} = -q \} = \langle i, j, k \rangle_{\mathbb{R}}$

$$Ad : Sp(1) \ni p \longmapsto (Ad(p) : q \mapsto pqp^{-1})$$

$$\in SO(3)$$

$\because)$ $\quad Ad(p) : \operatorname{Im} \mathbb{H} \longrightarrow \operatorname{Im} \mathbb{H}$ linear isom

$$|pqp^{-1}| = |p||q||p|^{-1} = |q|$$

$$\therefore \quad Ad(p) \in O(3)$$

$Sp(1)$ conn より $\quad Ad(Sp(1)) \subset SO(3)$ //

$\ker Ad = \{1, -1\}$

$\because)$ $\quad p \in \ker Ad \iff p q \bar{p} = q \quad \forall q$

$\iff p i \bar{p} = i,\ p j \bar{p} = j,\ p k \bar{p} = k$

$\iff p = \pm 1$ //

Ad : surj

$\because)$ $\quad Ad(e^{it}) i = e^{it} i e^{-it} = i$

$$Ad(e^{it})(aj + bk) = e^{2it}(aj + bk)$$

$$= (a\cos 2t - b\sin 2t) j + (a \sin 2t + b\cos 2t) k$$

$Ad(e^{it})$ は i 軸まわりの $2t$ 回転 —(a)

$Ad(e^{kt})$ は k 〃 —(b)

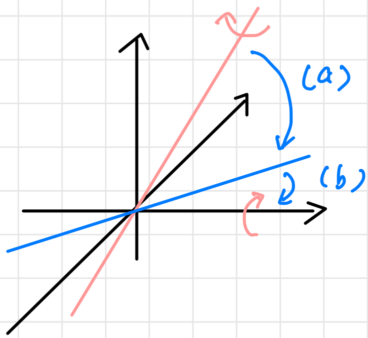
$g \in SO(3)$ は $\exists$軸 $\ell(g)$ まわりの回転

$\therefore$ $g =$ (a). (b) の形のいくつかの積

$g = Ad(\ )Ad(\ )\cdots$

$\quad = Ad(\exists h)$

//



以上から

$Ad : Sp(1) \cong SU(2)\ (\cong S^3) \longrightarrow SO(3)\ (\cong \mathbb{R}P^3)$

$\pi_1(S^3) = 1 \qquad\qquad \pi_1(SO(3)) = \mathbb{Z}_2$

Spin(n) in Cl_n

Thm

$\subset Cl_n^0$

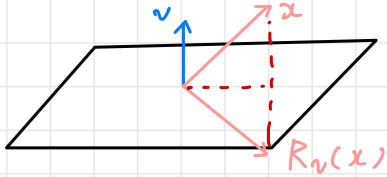
$Spin(n) = \{ g = v_1 \cdots v_{2p} \mid |v_i| = 1,\ p = 0, 1, 2 \cdots \}$

proof $\quad v \in \mathbb{R}^n \subset Cl_n \quad |v| = 1, \quad x \in \mathbb{R}^n$

このとき $\quad v^{-1} = -v/|v|^2 = -v \qquad v\cdot(-v) = +|v|^2 = 1$

$v x v^{-1} = -(xv - 2\langle x, v\rangle)v = -x + 2\langle x, v\rangle$

$$= -R_v(x)$$



$$g = v_1 \cdots v_{2p}$$

$$\begin{aligned} Ad(g)x &= gxg^{-1} \\ &= v_1 \cdots v_{2p}\, x\, v_{2p} \cdots v_1 \\ &= R_{v_1} \circ \cdots \circ R_{v_{2p}}(x) \qquad \therefore Ad(g) \in SO(n) \end{aligned}$$

$G = \{ g = v_1 \cdots v_{2p} \mid |v_i| = 1,\ p = 0, 1, 2 \cdots \}$ とする

(G 群, $v \cdot v = -1$ $\therefore \pm 1 \in G$)

$Ad : G \longrightarrow SO(n)$ homo を得た

<u>$\ker = \{\pm 1\}$</u>

$\because$) $Ad(g) = I_n \iff gxg^{-1} = x \ (\forall x \in \mathbb{R}^n)$

$\iff gx = xg$ (〃)

$\iff g\varphi = \varphi g \ (\forall \varphi \in Cl_n)$

g は <u>center</u> かつ in Cl_n^0

$\mathbb{R}\langle 1, \overset{\text{odd}}{\omega} \rangle$ for $n = 2m+1$

$\mathbb{R}\langle 1 \rangle$ for $n = 2m$

$\therefore g = c \in \mathbb{R} \qquad \therefore g = \pm 1$ //

<u>Ad : surj</u>

Fact (Cartan-Dieudonné)

For $g \in O(n)$, $\exists v_1, \cdots, v_p$ $(p \leqq n)$

s.t. $g = R_{v_1} \circ \cdots \circ R_{v_p}$ $\quad$ (p = even, $g \in SO(n)$) //

この Fact より, Ad surj //

$$1 \to \mathbb{Z}_2 \to G \xrightarrow{Ad} SO(n) \to 1 \quad \text{exact}$$

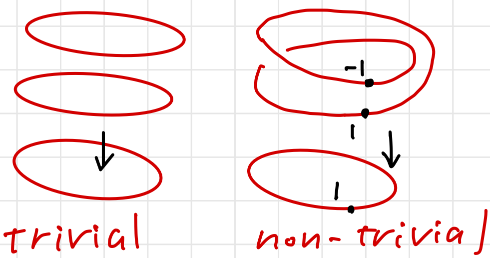
$$\rightsquigarrow \quad 1 \to \pi_1(G) \to \underset{\underset{\mathbb{Z}_2}{\shortparallel}}{\pi_1(SO(n))} \quad \text{exact}$$

$$\therefore \quad \pi_1(G) \cong \mathbb{Z}_2 \text{ or } \{1\}$$

double covering Ad が nontrivial を示せば

$\pi_1(G) = 1$ (G simply conn) & G conn が

わかり $G = Spin(n)$ となる



ここで

$$v_\pm := \cos\frac{t}{2} e_1 \pm \sin\frac{t}{2} e_2$$

$$|v_\pm| = 1$$

とおくと $\quad g(t) = -v_+ \cdot v_- \quad \in G$

$$= \cos t + \sin t \, e_1 e_2$$

$$g(0) = 1, \quad g(\pi) = -$$

$\therefore$ Ad : non trivial double cover $\quad \therefore$ $G = Spin(n)$ //

Note $n=1$ $SO(1)=1$, $Spin(1):=\{\pm 1\}$

$n=2$ $SO(2)=\left\{\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix} \middle| 0\leqq t<2\pi\right\}\cong S^1$

$Spin(2):=\{\cos t+\sin t\, e_1e_2 \mid 0\leqq t<2\pi\}\cong S^1$

$Ad: Spin(2)\to SO(2)$ double cover.

ex $n=4$

$Cl_4\cong \mathbb{H}(2)=\left\{\begin{pmatrix}p & q\\ r & s\end{pmatrix} \middle| p.q.r.s\in\mathbb{H}\right\}$

$e_1=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$, $e_2=\begin{pmatrix}0 & j\\ j & 0\end{pmatrix}$, $e_3=\begin{pmatrix}0 & k\\ k & 0\end{pmatrix}$

$e_4=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

$\therefore$ $Spin(4)=\left\{\begin{pmatrix}p & 0\\ 0 & q\end{pmatrix} \middle| p, q\in\mathbb{H}, |p|=|q|=1\right\}$

$= Sp(1)\times Sp(1)$

$\mathbb{H}\cong\mathbb{R}^4$ $\langle x.y\rangle := Re(x\bar{y})$ $(x.y\in\mathbb{H})$

$Ad: Spin(4)\ni(p.q)\longmapsto(x\mapsto pxq^{-1})\in SO(4)$

Note $|pxq^{-1}|=|x|$

$Ker\, Ad=\{(1.1),(-1,-1)\}$

$\therefore$ $SO(4)=Sp(1)\times Sp(1)/\mathbb{Z}_2 = SU(2)\times SU(2)/\mathbb{Z}_2$

Spinor space

G : Lie group

V : cpx vector space　$\dim < \infty$

$GL(V) = \{F : V \to V \mid F \text{ linear isom}\}$

$\cong GL(n:\mathbb{C})$

Def　Lie gr homo $\rho : G \to GL(V)$

(1) $(\rho . V)$ を G の V 上の representation 表現 という

(2) さらに V が Herm inner product $\langle \, , \rangle$

s.t. $\langle \rho(g) u, \rho(g) v \rangle = \langle u . v \rangle$

$(\forall u . v \in V, \quad \forall g \in G)$

のとき $(\rho . V)$ を unitary rep という.

Ex　$G = SO(3) \ni X$

$G \ni X \mapsto X \in GL(3:\mathbb{C})$　natural rep on $\mathbb{C}^3$

Ex　$G = SU(2) \ni X$

$G \ni X \mapsto X \in GL(2:\mathbb{C})$ natural rep on $\mathbb{C}^2$

Ex　$Ad : SU(2) \longrightarrow SO(3) \subset GL(3:\mathbb{C})$

adjoint rep on $\mathbb{C}^3$

$\rho : SO(3) \to GL(n; \mathbb{C})$

$\rightsquigarrow \rho \circ Ad : SU(2) \to GL(n; \mathbb{C})$

一方, $SU(2)$ の natural rep on $\mathbb{C}^2$ は

$SO(3)$ の rep にはならない

$Spin(3) = SU(2)$ とすると $(\rho, \mathbb{C}^2)$ が

spinor rep となる

$\rightsquigarrow$ $Spin(n)$ の natural rep (s.t. $SO(n)$ の rep

にはならないもの) を作ろう!!

$$\rho : Cl_{2m} \cong \mathbb{C}(2^m) \ni X \mapsto X \in End(\mathbb{C}^{2^m})$$

$$\rho_\pm : Cl_{2m+1} \cong \mathbb{C}(2^m) \oplus \mathbb{C}(2^m) \ni (X, Y)$$
$$\mapsto \begin{matrix} X \\ Y \end{matrix} \in End(\mathbb{C}^{2^m})$$

は $\mathbb{C}$ 上 alg homo (つまり, Cl_n の rep)

これらを $Spin(n) \subset Cl_n^0 \subset Cl_n$ へ

restrict する (natural rep)

Def

$$\Delta_{2m} := \rho|_{Spin(2m)}, \quad W_{2m} := \mathbb{C}^{2^m}$$

$$\Delta_{2m+1} := \rho_+|_{Spin(2m+1)}, \quad W_{2m+1} := \mathbb{C}^{2^m}$$

(Δ_n, W_n) を $Spin(n)$ の spinor rep という

W_n を spinor space という

Rem $\quad \rho_+ \not\cong \rho_-$ as Cl_{2m+1}-module

$\rho_+ \cong \rho_-$ as $Spin(2m+1)$-module

Rem $\quad -1 \in Spin(n)$ は -1 で act

$Ad(-1) = 1$

$\therefore$ Δ_n は $SO(n)$ の rep になりえない.

Ex $\quad Cl_3 \cong \mathbb{H} \oplus \mathbb{H}$, $\quad \mathbb{C}l_3 \cong \mathbb{C}(2) \oplus \mathbb{C}(2)$

$Spin(3) \cong Sp(1) \cong SU(2)$

$= \{(p, p) \mid p \in Sp(1)\}$ in $\mathbb{H} \oplus \mathbb{H}$

$= \{(A, A) \mid A \in SU(2)\}$ in $\mathbb{C}(2) \oplus \mathbb{C}(2)$

$\therefore$ $\rho_+(A, A) = A = \rho_-(A, A) \qquad \rho_+ \cong \rho_-$

$W_3 = \mathbb{C}^2$

$\Delta_3 : SU(2) \ni A \mapsto A \in GL(W_3) = GL(3 : \mathbb{C})$ //

Ex $Cl_4 \cong \mathbb{H}(2)$, $\mathbb{C}l_4 \cong \mathbb{C}(4)$

$$\begin{aligned} Spin(4) &= Sp(1) \times Sp(1) \\ &= \left\{ \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \middle| \ p, q \in Sp(1) \right\} \text{ in } \mathbb{H}(2) \\ &= \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| \ A, B \in SU(2) \right\} \text{ in } \mathbb{C}(4) \end{aligned}$$

$\therefore \quad W_4 = \mathbb{C}^4$

$$\Delta_4 : Spin(4) \ni \underbrace{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}}_{g\,''} \longmapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in GL(W_4) = GL(4 : \mathbb{C})$$

$\Delta_4(g) = A$, $\Delta_4(g) = B$ も rep on $\mathbb{C}^2$

$\rightsquigarrow \quad \Delta_4 = \Delta_4^+ \oplus \Delta_4^-$ irreducible decomp

Def (ρ, V), (ρ', V') rep of G are equivalent

$\underset{def}{\iff}$ $\exists\, \Phi : V \to V'$ linear isom s.t.

$$\begin{tikzcd} V \arrow[r, "\Phi"] \arrow[d, "\rho(g)"'] \arrow[dr, phantom, "\circlearrowright"] & V' \arrow[d, "\rho'(g)"] \\ V \arrow[r, "\Phi"'] & V' \end{tikzcd} \quad (\forall g \in G)$$

つまり、V と V' は G-module として同じもの。

Def (ρ, V) rep of G が irreducible

$\underset{\text{def}}{\Longleftrightarrow}$ 「 $W \subset V$ G-submodule

$\Rightarrow W = \{0\}$ or V 」

$(\rho(g) W \subset W, \ \forall g \in G)$

Fact G cpt Lie gr

(ρ, V) rep of G

$\Rightarrow \rho \cong \rho_1 \oplus \cdots \oplus \rho_k$ $\quad \rho_j$: irr

$(V = V_1 \oplus \cdots \oplus V_k)$

つまり、任意の rep は irr rep へ分解できる.

Def $(\rho_1, V_1), (\rho_2, V_2)$ rep of G

$(\rho_1 \oplus \rho_2)(g) := (\rho_1(g), \rho_2(g))$ on $V_1 \oplus V_2$

direct sum rep $(\rho_1 \oplus \rho_2, V_1 \oplus V_2)$

$(\rho_1 \otimes \rho_2)(g) = \rho_1(g) \otimes \rho_2(g)$ on $V_1 \otimes V_2$

tensor rep $(\rho_1 \otimes \rho_2, V_1 \otimes V_2)$

$\langle \rho_1^*(g)(f), v \rangle = \langle f, \rho_1(g^{-1}) v \rangle$

$\forall f \in V_1^*, \ \forall v \in V_1$

dual rep (ρ_1^*, V_1^*)

Thm

(1) $\Delta_{2m+1} = \rho_\pm|_{Spin(2m+1)}$ は irr rep

(2) $\Delta_{2m} = \rho|_{Spin(2m)}$

$\cong \Delta_{2m}^{+} \oplus \Delta_{2m}^{-}$

$(W_{2m} = W_{2m}^{+} \oplus W_{2m}^{-})$

$\Delta_{2m}^{\pm}$ は irr rep , $\Delta_{2m}^{+} \not\cong \Delta_{2m}^{-}$

$\omega_{\mathbb{C}}$ acts by ∓ 1 on $W_{2m}^{\pm}$

(3) $Cl_n, \mathbb{C}l_n \curvearrowright W_n$ Clifford multiplication

$n=2m$, $v \in \mathbb{R}^n \subset Cl_n$, $v \cdot W_{2m}^{\pm} \subset W_{2m}^{\mp}$

(4) W_n 上 $\exists$ Hermitian inne pr $\langle , \rangle$ s.t.

$\langle v\cdot\phi, \psi \rangle + \langle \phi, v\cdot\psi \rangle = 0 \quad \forall v \in \mathbb{R}^n, \ \forall \phi, \psi \in W_n$

特に Δ_n は $Spin(n)$ の unitary rep

i.e. $\langle \Delta(g)\phi, \Delta(g)\psi \rangle = \langle \phi, \psi \rangle$ //

outline of proof

(1) $n=2m+1$ $\mathbb{C}l_n = \mathbb{C}l_n^{+} \oplus \mathbb{C}l_n^{-} = \mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$

$Spin(n) \subset \mathbb{C}l_n^{0} = \{(A, A) \mid A \in \mathbb{C}(2^m)\}$

$\therefore \quad \rho_+|_{Spin(n)} \cong \rho_-|_{Spin(n)}$

$\rho_\pm|_{Cl_n^0} : \underset{\mathbb{C}(2^m)}{\overset{}{Cl_n^0}} \ni (A.A) \longmapsto A \in \mathbb{C}(2^m)$

は Cl_n^0 の irr rep である (see テキスト)

Cl_n^0 は $\{e_i e_j\}$ で生成されるが,

$\{e_i e_j\} \subset Spin(n)$

そこで $\Delta_n = \rho_\pm|_{Spin(n)}$ not irr とすると

$\rho_\pm|_{Cl_n^0}$ も Cl_n^0 の rep として not irr 矛盾。

(2) $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ $\quad e_1, \cdots, e_n, e_{n+1}$ o.n.b.

$Cl_n \ni v \longmapsto v e_{n+1} \in Cl_n^0$ により $Cl_n \cong Cl_{n+1}^0$ as alg

とくに $\underset{\cup \atop Spin(2m)}{Cl_{2m}^0} \cong Cl_{2m-1} \cong \mathbb{C}(2^{m-1}) \oplus \mathbb{C}(2^{m-1})$

$\rho|_{Cl_{2m}^0} = \rho_+ \oplus \rho_-$ as Cl_{2m}^0-module

$\therefore \Delta_{2m} = \rho|_{Spin(2m)} = \Delta_{2m}^+ \oplus \Delta_{2m}^-$

また, $\omega_{\mathbb{C}} = (1. -1)$ in $\mathbb{C}(2^{m-1}) \oplus \mathbb{C}(2^{m-1})$ より

$\omega_{\mathbb{C}} = \pm 1$ on $W_{2m}^\pm$

$\Delta_{2m}^\pm$ irr, $\Delta_{2m}^+ \not\cong \Delta_{2m}^-$ は (1) と同様。

(3) $n=2m$, $v\cdot\omega_{\mathbb{C}}=-v\omega_{\mathbb{C}}$ より $vW_{2m}^{\pm}\subset W_{2m}^{\mp}$

(4) $Cl_2 \ni 1, e_1, e_2, e_1e_2$

$$\underset{\cong}{\longmapsto} \begin{pmatrix}1&0\\0&1\end{pmatrix}, \sqrt{-1}\sigma_1, \sqrt{-1}\sigma_2, (-)\sqrt{-1}\sigma_3 \in \mathbb{C}(2)$$

$(\sqrt{-1}\sigma_i)^* + \sqrt{-1}\sigma_i = 0$ に注意すると

$Cl_2 \cong \mathbb{C}(2) \curvearrowright \mathbb{C}^2$ であるが

$$\langle v\cdot\varphi, \psi\rangle + \langle\varphi, v\cdot\psi\rangle = 0$$

$$\forall \varphi, \psi \in \mathbb{C}^2 = W_2, \ \forall v \in \mathbb{R}^2$$

また

$$Cl_{2m} \cong Cl_2 \otimes \cdots \otimes Cl_2 \quad \text{as alg}$$

$$\mathbb{R}^{2m} \ni e_{2i-1} \longmapsto 1\otimes\cdots\otimes 1\otimes \overset{i}{\sqrt{-1}\sigma_1}\otimes 1\otimes\cdots\otimes 1$$

$$e_{2i} \longmapsto 1\otimes\cdots\otimes 1\otimes \overset{i}{\sqrt{-1}\sigma_2}\otimes 1\otimes\cdots\otimes 1 \qquad i=1,2,\cdots,m$$

なので

$$Cl_{2m} \curvearrowright \mathbb{C}^2\otimes\cdots\otimes\mathbb{C}^2 = \mathbb{C}^{2^m} = W_{2m} !!$$

tensor inner product on $\mathbb{C}^{2^m}$ を入れると,

$$\langle v\cdot\varphi, \psi\rangle + \langle\varphi, v\cdot\psi\rangle = 0 \quad (v\in\mathbb{R}^n)$$

$|v|=1$ なら,

$$\langle v\cdot\varphi, v\cdot\psi\rangle = -\langle \underbrace{v\cdot v}_{=-1}\cdot\varphi, \psi\rangle = \langle\varphi,\psi\rangle$$

$\therefore$ $g\in Spin(n)$ $\langle g\varphi, g\psi\rangle = \langle\varphi,\psi\rangle$ //

§ vector bundle and principal bundle

Def M n-dim mfd

$p: E \to M$ real vector bundle on M
rank $= m$

(1) E $(m+n)$-dim mfd, p : smooth surj

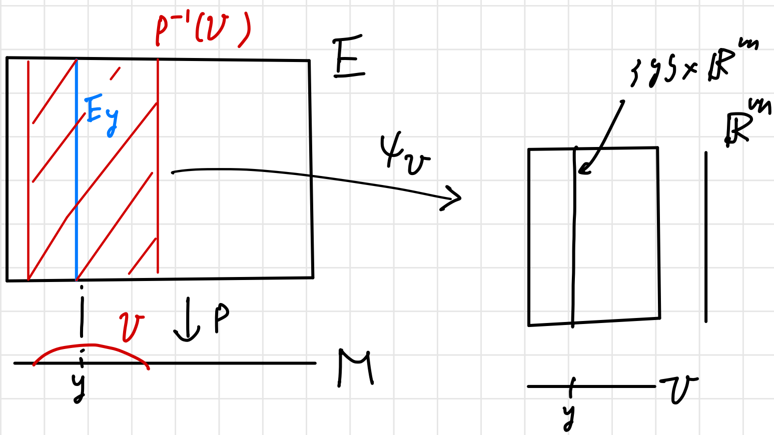
(2) $E_x := p^{-1}(x)$ is m-dim vector sp
← fiber of E on x

(3) (local triviality)

$\forall x \in M, \ ^\exists U$: x の nbd. $^\exists \psi_U : p^{-1}(U) \longrightarrow U \times \mathbb{R}^m$

s.t. • ψ_U diffeo

• $\psi_U|_{E_y} : E_y \to \{y\} \times \mathbb{R}^m$ liner isom
$(\forall y \in U)$



rank$=1$ のとき line bundle という

また，$s : M \longrightarrow E$ smooth map $p \circ s = id_M$

を **E の section（切断）** といい

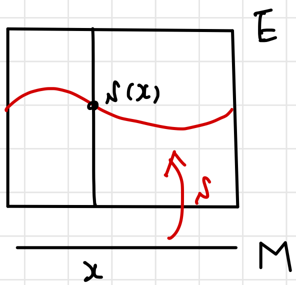
$$\underline{\Gamma(M, E)} := \{ s \mid s : M \to E \text{ section} \}$$

For $s_1, s_2 \in \Gamma(M, E)$ and $f \in C^\infty(M)$,

$$(s_1 + s_2)(x) := s_1(x) + s_2(x)$$

$$(f s_1)(x) := f(x) s_1(x)$$

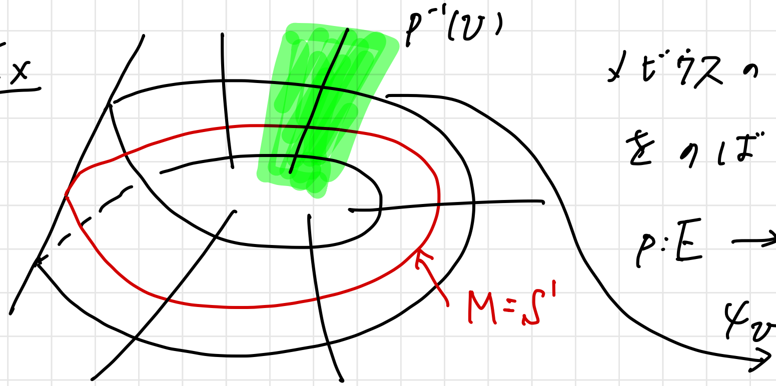
により $\Gamma(M, E)$ は $C^\infty(M)$-module



$p \circ s(x) = x$

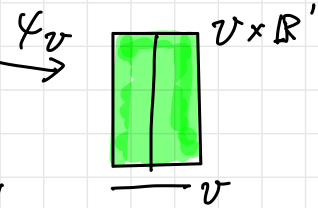
$\therefore \ s(x) \in p^{-1}(x) = E_x \quad (^\forall x \in M)$

Ex

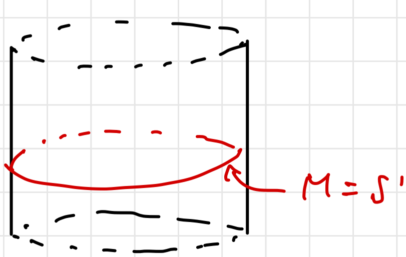


メビウスの帯のタテ方向をのばす

$p : E \longrightarrow M = S^1$



Ex



$$E = S^1 \times \mathbb{R}^1$$
$$\downarrow p$$
$$M = S^1$$

Ex M n-dim mfd

$$p: TM := \bigcup_{x \in M} T_x M \to M \quad \text{tangent bundle}$$

$\mathfrak{X}(M) := \Gamma(M, TM)$ M上 vector fields

$$p: T^*M := \bigcup_{x \in M} T_x^* M \to M \quad \text{cotangent bundle}$$

$\Omega^1(M) := \Gamma(M, T^*M)$ M上 1-forms

transition function for $E \to M$

$p: E \to M$ rank m vector bundle

$$\psi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^m$$

$$\psi_\beta : p^{-1}(U_\beta) \to U_\beta \times \mathbb{R}^m$$

$\rightsquigarrow$ $\psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^m \to (U_\alpha \cap U_\beta) \times \mathbb{R}^m$

各 fiber 上で linear isom

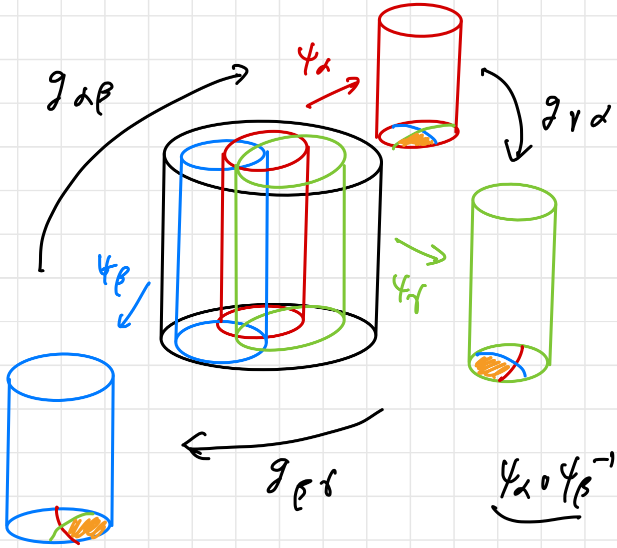
$\therefore$ $\psi_\alpha \circ \psi_\beta^{-1}(x, v) = (x, {}^{\exists}g_{\alpha\beta}(x) v)$

$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(m; \mathbb{R})$

transition fn という

lem

$$\begin{cases} g_{\alpha\alpha} = I_m \text{ on } U_\alpha \\ g_{\alpha\beta} = g_{\beta\alpha}^{-1} \quad \text{on } U_\alpha \cap U_\beta \\ g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = I_m \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \neq \phi \end{cases}$$



$$\underbrace{\psi_\alpha\circ\psi_\beta^{-1}}\circ\underbrace{\psi_\beta\circ\psi_\gamma^{-1}}\circ\underbrace{\psi_\gamma\circ\psi_\alpha^{-1}} = id$$

逆に、 $M = \bigcup_{\alpha\in A} U_\alpha$ open covering

& $\{g_{\alpha\beta}\}_{\alpha,\beta\in A}$ $\left(g_{\alpha\beta}: U_\alpha\cap U_\beta \to GL(m;\mathbb{R})\right)$

satisfying cocycle condition

$\Rightarrow$ $\{U_\alpha\times\mathbb{R}^m\}_{\alpha\in A}$ を $\{g_{\alpha\beta}\}$ で張り合わせる

と vector bundle E を得る.

<u>proof</u> $\tilde{E} = \coprod_{\alpha\in A} U_\alpha\times\mathbb{R}^m$

$(x,v)\in U_\alpha\times\mathbb{R}^m \sim (y,w)\in U_\beta\times\mathbb{R}^m$ を

$x=y,\quad v = g_{\alpha\beta}(x)\,w$ とする

このとき $E := \tilde{E}/\sim$ は mfd であり

$p: E\to M$ vector bundle となる //

もちろん　$E \rightsquigarrow \{g_{\alpha\beta}\} \rightsquigarrow \tilde{E}/_{\sim} \cong E$

bundle isom

ここで　$p_1 : E \to M$,　$p_2 : F \to M$

が bundle isom　$E \cong F$ とは

$$
\begin{tikzcd}
E \arrow[r] \arrow[dr] & F \arrow[d] \\
& M
\end{tikzcd}
$$

$^\exists \Phi : E \to F$　diffeo s.t.

$p_2 \circ \Phi = p_1$　&　$\Phi|_{E_x} : E_x \to F_x$

linear isom

<u>Exc</u>　$p : E \to M$　vector bundle

$M = \cup U_\alpha$,　$\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^m$

$\{g_{\alpha\beta}\}$　trans fn

$s \in \Gamma(M, E)$ に対し

$\varphi_\alpha(s(x)) = (x, s_\alpha(x))$ により

$\{s_\alpha : U_\alpha \to \mathbb{R}^m\}_{\alpha \in A}$ を得る.

このとき　$s_\alpha = g_{\alpha\beta} s_\beta$　を示せ

i.e.　section とは local $\mathbb{R}^m$-valued fn を

$\{g_{\alpha\beta}\}$ ではり合わせたもの //

Ex (orientation)

M n-dim mfd atlas $\{(U_\alpha, \varphi_\alpha)\}$

$U_\alpha : x_1, \cdots, x_n, \quad U_\beta : y_1, \cdots, y_n$

T^*M の tran fn $g_{\alpha\beta}(x) = \left(\frac{\partial y_i}{\partial x_j}\right)_{i,j}$

$$\Lambda^n T^*M = \bigcup_{x \in M} \Lambda^n(T_x^*M) \longrightarrow M$$

の tran fn は $\det g_{\alpha\beta} = \det\left(\frac{\partial y_i}{\partial x_j}\right)$

$\omega \in \Omega^n(M) = \Gamma(M, \Lambda^n T^*M)$

は $\omega = s_\beta \, dy_1 \wedge \cdots \wedge dy_n$ on U_β

$= \underbrace{s_\beta \det\left(\frac{\partial y_i}{\partial x_j}\right)}_{= s_\alpha} dx_1 \wedge \cdots \wedge dx_n$ on $U_\alpha \cap U_\beta$

Def M is orientable

$\exists \omega \in \Omega^n(M)$ s.t. $\omega_x \neq 0 \ (\forall x \in M)$

(つまり nowhere vanishing n-form が存在)

このとき、$\{$nowhere vanishing n-forms$\}/\sim \ = \{1, -1\}$

$\omega \sim \omega' \overset{def}{\Longleftrightarrow} \exists f \in C^\infty(M)$ s.t. $f > 0, \quad \omega = f\omega'$,

どちらかを正の向きとして、orientation が定まる。

また　$[\omega]$　orientation of M

$$\omega = f\, dx_1 \wedge \cdots \wedge dx_n, \quad f > 0$$

となる local coord $x_1, \cdots x_n$ を positive coordinate

という

$f < 0$ なら　$x_1 \to -x_1$ により positive coord となる.

$\therefore$ $\exists$ atlas $\{(U_\alpha, \varphi_\alpha)\}$ of M s.t.

$$\left.\begin{array}{l} \\ J(\varphi_\alpha \circ \varphi_\beta^{-1}) > 0 \quad (\forall \alpha, \beta) \end{array}\right\} ✡$$

$\because)$ $\quad \omega = \underbrace{s_\beta \det\left(\dfrac{\partial y_i}{\partial x_j}\right)}_{s_\alpha} dx_1 \wedge \cdots \wedge dx_n$

$s_\alpha > 0$, $s_\beta > 0$ より $\det\left(\dfrac{\partial y_i}{\partial x_j}\right) > 0$ //

逆に ✡ が成立するなら

U_α 上　$\omega_\alpha = dx_1 \wedge \cdots \wedge dx_n$

$\{\rho_\alpha\}$　1の分割

$\omega := \sum \rho_\alpha \omega_\alpha$ は nowhere vanishing //

tangent frame bundle

目的: $TM \rightsquigarrow GL(M)$ pr $GL(n;\mathbb{R})$-bundle

M : n-dim mfd

$TM = \bigcup T_xM$ tangent bundle

$L_x := \{(X_1, \cdots, X_n) \mid X_1, \cdots, X_n$ T_xMの basis$\}$

<u>Def</u> $GL(M) := \bigcup_{x \in M} L_x$

$\pi : GL(M) \to M$ $\quad \pi^{-1}(x) = L_x$

を tangent frame bundle of M という.

<u>Lem</u> $GL(M)$ is $n+n^2$-dim mfd

∵) $\{(U_\alpha, \varphi_\alpha)\}$ atlas of M

$\varphi_\alpha(x) = (x_1, \cdots, x_n)$ local coord

$(X_1, \cdots, X_n) \in L_x$ に対し

$(X_1, \cdots, X_n) = \left(\left(\frac{\partial}{\partial x_1}\right), \cdots, \left(\frac{\partial}{\partial x_n}\right)\right) {}^{\exists !}g$

$g \in GL(n;\mathbb{R})$ を得る

$\pi^{-1}(U_\alpha) = \bigcup_{x \in U_\alpha} L_x \ni u = (x, (X_1, \cdots, X_n))$

$\xmapsto[\text{bij}]{} (\varphi_\alpha(x), g) \in \varphi_\alpha(U_\alpha) \times GL(n;\mathbb{R}) \underset{\text{open}}{\subset} \mathbb{R}^n \times \mathbb{R}(n)$

coord change も smooth //

$\exists$ free right action of $GL(n;\mathbb{R})$

$GL(M) \curvearrowleft GL(n;\mathbb{R})$ s.t.

$\forall x \in M$, $L_x \circlearrowleft GL(n;\mathbb{R})$ free & transitive

G-action の復習

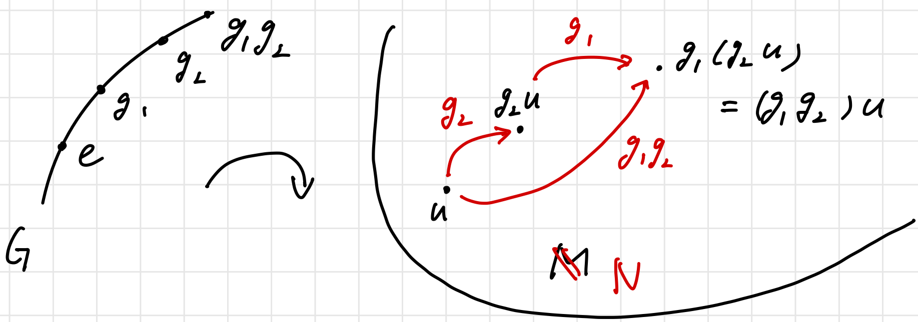
N mfd, G Lie group

$G \curvearrowright N$ left action of G on N とは

$\Phi : G \times N \ni (g, u) \mapsto gu \in N$ smooth

s.t. $(g_1 g_2)u = g_1(g_2 u)$, $eu = u$

$\forall g_1, g_2 \in G$, $u \in \cancel{M}\ N$



$u \in N$: isotropy group at u とは

$$G_u := \{ g \in G \mid gu = u \} \subset G \quad \text{subgr}$$

G-orbit at u とは $G \cdot u = \{ gu \mid g \in G \} \subset N$

(このとき $G/G_u \ni [g] \mapsto gu \in G\cdot u$ (diffeo))

- $G \curvearrowright N$ free とは. $^{\forall}u \in N,\ G_u = \{e\}$

  $(\text{i.e. } G\cdot u \cong G \quad ^{\forall}u \in N)$

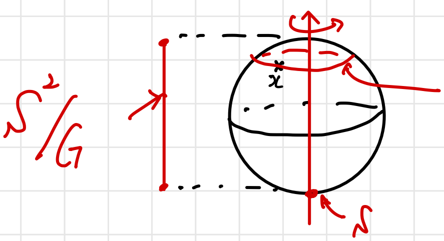
- $G \curvearrowright N$ transitive とは.

  $^{\forall}u, u' \in N,\ ^{\exists}g \in G \text{ s.t. } gu = u'$

  $(\text{i.e. } G \cdot u = M)$

- M/G orbit space (mfd とは限らない)

<u>Ex</u> $M = S^2$ $\quad G = SO(2) \curvearrowright S^2$ z軸回転



$G \cdot x \cong S^1 \cong SO(2)$

$G \cdot N = N \quad (G_N = SO(2))$

$G \cdot S = S \quad (G_S = SO(2))$

<u>Exc</u> $G = SO(3) \curvearrowright S^2$

- $SO(3) \times S^2 \ni (g, x) \longmapsto gx \in S^2$ transitive な作用であることを示せ

- $N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in S^2$, G_N を計算し $G_N \cong SO(2)$ を示せ

  (よって $S^2 = SO(3)/SO(2)$)

さて, $GL(M) \curvearrowleft GL(n; \mathbb{R})$ をつくろう.

$$u = (x, (X_1, \cdots, X_n)) \in L_x \subset GL(M)$$

$GL(M) \times GL(n:\mathbb{R}) \ni (u, g)$

$\mapsto ((x, (X_1, \cdots, X_n)g)) \in GL(M)$

そして

u の orbit $\{ug \mid g \in GL(n:\mathbb{R})\} = L_x$

$\therefore$ L_x へ $GL(n:\mathbb{R})$, free transitive //

さらに, この action について.

$GL(M)$ の local triviality は

$GL(n:\mathbb{R})$-equivalent (同変)

$\because)$

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \ni u \arrow[r, mapsto, "\Phi_\alpha"] \arrow[d, "g'"] \arrow[dr, phantom, "\circlearrowright"] & (x, g) \in U_\alpha \times GL(n:\mathbb{R}) \arrow[d, "g'"] \\
\pi^{-1}(U_\alpha) \ni ug' \arrow[r, mapsto] & (x, gg') \in U_\alpha \times GL(n:\mathbb{R})
\end{tikzcd}$$

$\forall g' \in GL(n:\mathbb{R})$ //

$g = \phi_\alpha(u)$ とすれば…

上は $\phi_\alpha(ug') = \phi_\alpha(u)g'$ とかける.

$\phi_\alpha(u)\phi_\beta(u)^{-1} = \phi_\alpha(ug')\phi_\beta(ug')^{-1}$ より,

$g_{\alpha\beta}(x) := \phi_\alpha(u)\phi_\beta(u)^{-1} : U_\alpha \cap U_\beta \to GL(n:\mathbb{R})$

を得る.

Exc この $g_{\alpha\beta}$ は TM の tran fn と一致

することを示せ.

$\therefore$ $GL(M)$ は $\{U_\alpha \times GL(n;\mathbb{R})\}_{\alpha\in A}$ を $\{g_{\alpha\beta}\}$

で はり合わせたもの //

pr G-bundle on M を def しておく.

<u>Def</u> G : Lie group, P, M : mfd

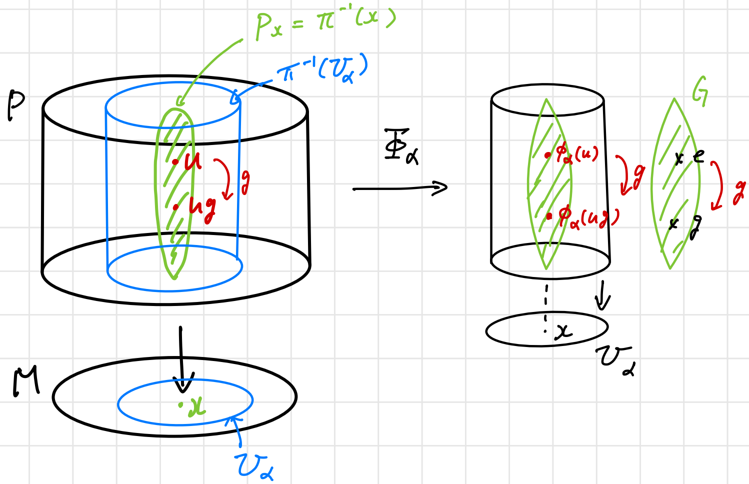
$\pi : P \to M$ が 構造群 (structure group) $= G$

の 主 G 束 (principal G-bundle) とは,

$P \curvearrowleft G$
$\downarrow$
M

- π : surj smooth
- $P \curvearrowleft G$, $P_x := \pi^{-1}(x)$ (fiber) は この作用で保存され, $P_x \curvearrowleft G$ は free & transitive
- G-eq local triviality が成立
  $\forall x \in M$, $^\exists U$ x の nbd
  $\exists$ diffeo
  $\Phi_U : \pi^{-1}(U) \ni u \longmapsto (\pi(u), \phi_U(u)) \in U \times G$
  s.t. $\phi_U(ug) = \phi_U(u) g$ //

Note
- $\pi^{-1}(x) \cong G$ (diffeo)
- P/G orbit space $= M$
- local triviality $\rightsquigarrow$ tran fn $\{g_{\alpha\beta}\}$
  $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$



<u>Ex</u> (M, g) oriented Riem mfd

$g \in \Gamma(M, T^*M \otimes T^*M)$ s.t. $\forall x \in M$

$g_x : T_xM \times T_xM \longrightarrow \mathbb{R}$ positive definite

locally $g = \sum g_{ij}\, dx_i \otimes dx_j$

$(g_{ij})_{i,j}$ p.d. symm matrix

$SO_x := \{ (X_1, \cdots, X_n) \mid$ positive. o.n.b of $T_xM \}$

$SO(M) := \bigcup_{x \in M} SO_x \curvearrowleft SO(n)$

oriented orthnormal frame bundle

<u>Note</u> $GL(M)$, $SO(M)$ の local section

は local frame とよぶ

Exc global section $s: M \to P$ が存在

$\iff$ $P \cong M \times G$ trivial pr G-bundle

ここで $P \cong P'$ とは

$F: P \to P'$ diffeo s.t.

$\pi' \circ F = \pi$ & $F(ug) = F(u)g$

$\forall u \in P, \ \forall g \in G$ //

Ex

$S^3 = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1 \}$

$\pi: S^3 \ni (z_1, z_2) \mapsto [z_1 : z_2] \in \mathbb{C}P^1$

$S^3 \curvearrowleft U(1)$ $\quad (z_1, z_2) e^{i\theta} := (e^{i\theta} z_1, e^{i\theta} z_2)$

$$\begin{array}{c} S^3 \\ \downarrow U(1) = S^1 \\ S^2 \end{array}$$

pr $U(1)$-bundle

Hopf bundle //

Ex

$S^7 = \{ (p, q) \in \mathbb{H}^2 \mid |p|^2 + |q|^2 = 1 \}$

$S^7 \ni (p, q) \longmapsto [p : q] \in \mathbb{H}P^1 = S^4$

$S^7 \curvearrowleft Sp(1) = SU(2) = S^3$

$$\begin{array}{c} S^7 \\ \downarrow S^3 \\ S^4 \end{array}$$

pr $SU(2)$-bundle

(instanton bundle) //

vector bundle associated to pr G-bundle

$\pi : P \to M$ pr G-bundle

$G \underset{\rho}{\curvearrowright} V$ rep of G

このとき $P \times V \curvearrowleft G$ を

$(u, v)g := (ug, \rho(g^{-1})v)$ とすると

$P \times_\rho V := P \times V / G$

$p : P \times_\rho V \ni [u, v] \mapsto \pi(u) \in M$

この $P \times_\rho V$ は M 上の vector bundle

s.t. $p^{-1}(x) \cong V$

vector bundle associated to P

同伴ベクトル束 という

つまり $\{U_\alpha \times V\}_\alpha$ を $\{\rho(g_{\alpha\beta})\}_{\alpha,\beta}$ で はり合わせた bundle.

($\rho(g_{\alpha\beta}) : U_\alpha \cap U_\beta \xrightarrow{g_{\alpha\beta}} G \xrightarrow{\rho} GL(V)$)

Ex $P = GL(M)$, $GL(n;\mathbb{R}) \curvearrowright^{\iota} \mathbb{R}^n$

$\iota(g) = g$

$\rightsquigarrow$ $TM \cong P \times_{\iota} \mathbb{R}^n$

$\iota^*(g) = {}^t g^{-1}$

$\rightsquigarrow$ $T^*M \cong P \times_{\iota^*} \mathbb{R}^n$

Ex

$O(M) \curvearrowleft O(n)$

$\iota(g) = g = {}^t g^{-1} = \iota^*(g)$

$\therefore$ $TM \cong T^*M$

これは. $T_x M \ni X_x \longmapsto g(X_x, \cdot) \in T_x^* M$

による

G-inv element → Section

α : G-inv element in V $(\rho(g)\alpha = \alpha)$

$\Rightarrow$ $s \in \Gamma(M, P \times_{\rho} V)$

def by $s(x) = [u, \alpha]$

$\because)$ $[ug, \alpha] = [u, \rho(g)\alpha] = [u, \alpha]$

$\therefore$ $s(x)$ は well defined

Ex (M, g) Riem mfd

$d = \sum e_i \otimes e_i \in \mathbb{R}^n \otimes \mathbb{R}^n$ $\{e_i\}$ o.n.b

$O(n)$-inv element

$\rightsquigarrow g = [u, d] \in T^*M \otimes T^*M$ Riem metric

Ex (M, g) oriented Riem

$SO(M) \curvearrowleft SO(n)$

$\det : SO(n) \ni h \longmapsto 1 \in GL(1; \mathbb{R})$

$1 \in \mathbb{R}$ is inv element

$\Omega \in \Omega^n(M)$ $\Lambda^n T^*M \cong M \times \mathbb{R}$

$\Omega = \omega_1 \wedge \cdots \wedge \omega_n$ $\{\omega_i\}$ dual of o.n.f $\{e_i\}$

$= \sqrt{|g_{ij}|}\, dx_1 \wedge \cdots \wedge dx_n$ in p. local coord

G-eq str on V $\Rightarrow$ associated str on $P \times_\rho V$

Ex $\begin{matrix} P \\ \downarrow G \\ M \end{matrix}$ pr G-bundle

$G \curvearrowright V$
$\langle u, u' \rangle = \langle \rho(g) u, \rho(g) u' \rangle$ $\forall g \in G$

V unitary rep of G

$\rightsquigarrow P \times_\rho V \ni$ fiber metric

$\langle [u,\varphi],[u,\psi]\rangle_x := \langle \varphi,\psi\rangle \qquad \pi(u)=x$

とすると (well defined) Herm inner product on $p^{-1}(x) = (P\times_\rho V)_x$ //

<u>Ex</u> (M,g) : Riem mfd

$$\begin{array}{c} O(M) \curvearrowleft O(n) \\ \downarrow \\ M \end{array}$$

$$\underset{\substack{\rotatebox{90}{$\in$} \\ h}}{O(n)} \underset{\nu}{\curvearrowright} \underset{\substack{\rotatebox{90}{$\in$} \\ x}}{\mathbb{R}^n} \qquad \begin{tikzcd} x \arrow[r, mapsto] & hx \\ \mathbb{R}^n \arrow[r] \arrow[d, hook] & Cl_n \\ Cl_n \arrow[ur] & \end{tikzcd}$$

$(hx)^2 = -\langle hx, hx\rangle = -\langle x,x\rangle = -\|x\|^2$

$\therefore$ h extends to alg homo on Cl_n

by $h(v_1\cdots v_p) = (hv_1)\cdots(hv_p)$

alg homo = Clifford product は $O(n)$-eq

よって $Cl(M) := O(M)\times_\nu Cl_n$

$\mathbb{C}l(M) := O(M)\times_\nu \mathbb{C}l_n$

に fiber 上に Clifford product str が入る

$[u,\varphi][u,\psi] := [u,\varphi\cdot\psi]$

よって

$\varphi, \psi \in \Gamma(M, Cl(M))$

$$(\varphi \cdot \psi)(x) := \varphi(x) \cdot \psi(x) \in \Gamma(M, Cl(M))$$ //

$$Cl(M) = Cl^0(M) \oplus Cl^1(M)$$

$$= Cl^+(M) \oplus Cl^-(M) \quad (n = 2m)$$

$$\omega_{\mathbb{C}} = (\sqrt{-1})^{[\frac{n+2}{2}]} e_1 \cdots e_n \in \Gamma(M, Cl(M))$$

などでも成立

$\because)$ $Cl_n = Cl_n^+ \oplus Cl_n^- \leftarrow O(n)$-eq //

次に spinor space W を fiber とする vector bundle を作りたい。

$$\begin{tikzcd} \mathrm{Spin}(n) \arrow[r, "\Delta"] \arrow[d, "\text{double cover}"'] & U(W) \\ SO(n) \arrow[ru, "\times"'] & \end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd} & \boxed{\mathrm{Spin}(M)} \arrow[dl] \arrow[d] & \mathrm{Spin}(n) \arrow[l, phantom, "\curvearrowleft"] \arrow[d] \\ M & SO(M) \arrow[l] & SO(n) \arrow[l, phantom, "\curvearrowleft"] \end{tikzcd}$$

左のような pr Spin(n)-bundle を作るべき

§ Spin structure

$$\begin{array}{c} P \\ \downarrow G \\ M \end{array}$$

local triviality

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \ni u \longrightarrow (\pi(u), \varphi_\alpha(u)) \in U_\alpha \times G$$

$$g_{\alpha\beta}(x) = \varphi_\alpha(u)\varphi_\beta(u)^{-1}$$

$\rightsquigarrow$ $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ s.t.

cocycle condition $g_{\alpha\alpha} = I$, $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = I$$

別の trivialization, $f_\alpha : U_\alpha \to G$

$$\pi^{-1}(U_\alpha) \ni u \longmapsto (x, f_\alpha(x)\varphi_\alpha(u)) \in U_\alpha \times G$$

$$g_{\alpha\beta}'(x) = \varphi_\alpha'(u)\varphi_\beta'(u)^{-1}$$
$$= f_\alpha(x) g_{\alpha\beta}(x) f_\beta(x)^{-1}$$

$\{g_{\alpha\beta}\}$, $\{g_{\alpha\beta}'\}$ は同じ P を与える

P, Q : 主 G 束 on M

$P \rightsquigarrow$ $\mathcal{U} = \{U_{\alpha'}\}_{\alpha' \in A'}$ で局所自明化

$Q \rightsquigarrow$ $\mathcal{V} = \{V_{\alpha''}\}_{\alpha'' \in A''}$ 〃

$\mathcal{U}$ と $\mathcal{V}$ に共通な細分 $\mathcal{W} = \{W_\alpha\}_{\alpha \in A}$ をとり,

$(\forall \alpha \in A, \exists \alpha' \in A' \ W_\alpha \subset U_{\alpha'})$

Φ_α', Φ_α'' を W_α へ restrict

$\rightsquigarrow$ P, Q は $\mathcal{W} = \{W_\alpha\}_{\alpha\in A}$ で局所自明化

P の tran fn $g_{\alpha\beta}$ 　 Q の tran fn $\{h_{\alpha\beta}\}$

$P \cong Q$ $\Leftrightarrow$ ${}^\exists\{f_\alpha : W_\alpha \to G\}_{\alpha\in A}$ s.t.

$h_{\alpha\beta}(x) = f_\alpha^{-1}(x)\, g_{\alpha\beta}(x)\, f_\beta(x)$ 　 $x \in U_\alpha \cap U_\beta$ //

そこで

M : mfd, 　 $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ open covering of M

$\{g_{\alpha\beta}\}_{\alpha,\beta}$ 　 $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ + cocycle cond.

$\{g_{\alpha\beta}\} \sim \{g'_{\alpha\beta}\} \overset{\text{def}}{\Leftrightarrow} {}^\exists\{f_\alpha\}_{\alpha\in A}$ s.t. $g'_{\alpha\beta} = f_\alpha^{-1} g_{\alpha\beta} f_\beta$

$[g_{\alpha\beta}] :=$ $\{g_{\alpha\beta}\}$ の同値類

$H^1(\mathcal{U}, \underline{G}) := \{ [g_{\alpha\beta}] \mid \{g_{\alpha\beta}\}$ as above $\}$

covering について "inductive limit" をとって

$H^1(M, \underline{G}) := \varinjlim H^1(\mathcal{U}, \underline{G})$

M 上 $\underline{G}$ 係数の 1次(チェック) コホモロジーという.

これまで見てきたことから

$\mathrm{Prin}_G(M) = \{ P \mid P$ は M 上主 G 束 $\}/\sim$ $\cong H^1(M, \underline{G})$

(主 G 束の同型類全体)

//

$\mathbb{Z}$ 係数チェックコホモロジー

M : mfd, $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ covering

$U_{\alpha_0 \cdots \alpha_p} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \neq \phi$ 上

$f_{\alpha_0 \cdots \alpha_p} : U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \to \mathbb{Z}$ locally const

$\{ f_{\alpha_0 \cdots \alpha_p} \mid \alpha_0, \cdots, \alpha_p \in A \}$

を $\mathbb{Z}$ 値 p-cochain という.

その全体を $C^p(\mathcal{U}, \mathbb{Z})$ とする

$$\delta : C^p(\mathcal{U}, \mathbb{Z}) \longrightarrow C^{p+1}(\mathcal{U}, \mathbb{Z})$$

$$f = \{f_{\alpha_0 \cdots \alpha_p}\} \longmapsto \delta f$$

$$(\delta f)_{\alpha_0 \cdots \alpha_{p+1}} := f_{\alpha_1 \cdots \alpha_{p+1}} - f_{\alpha_0 \alpha_2 \cdots \alpha_{p+1}} + \cdots + (-1)^{p+1} f_{\alpha_0 \cdots \alpha_p}$$

Exc $\delta^2 = 0$

$$0 \to C^0(\mathcal{U}, \mathbb{Z}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathbb{Z}) \to \cdots \to \quad \text{cochain cpx}$$

Def $H^p(\mathcal{U}, \mathbb{Z}) := H^p(C^*(\mathcal{U}, \mathbb{Z}))$

$$= \frac{\ker(\delta : C^p \to C^{p+1})}{\mathrm{Im}(\delta : C^{p-1} \to C^p)}$$

coveringの細分で inductive limit をとる

$$\check{H}^p(M,\mathbb{Z}) := \varinjlim H^p(\mathcal{U}.\mathbb{Z})$$

つまり，

$$\check{H}^p(M.\mathbb{Z}) := \bigcup_{\mathcal{U}:M\text{のcover}} H^p(\mathcal{U}.\mathbb{Z}) \Big/ \sim$$

$[f] \sim [g] \qquad f \in C^p(\mathcal{U}.\mathbb{Z}),\ g \in C^p(\mathcal{V}.\mathbb{Z})$

$\underset{\text{def}}{\Leftrightarrow}$ $\mathcal{U}$ と $\mathcal{V}$ に共通な細分 $\mathcal{W} = \{W_\beta\}_{\beta \in B}$

$[f]$ と $[g]$ の $W_{\beta_0 \cdots \beta_p}$ への restriction

が一致　　(← 正確なdefはテキスト)

<u>Ex</u>　$[f] \in \check{H}^0(M.\mathbb{Z})$

$\mathcal{U} = \{U_\alpha\}$,　$f_\alpha : U_\alpha \to \mathbb{Z}$　locally const

s.t.　$(\delta f)_{\alpha\beta} = f_\beta - f_\alpha = 0$　on $U_{\alpha\beta}$

$$U_{\alpha\beta} = U_\alpha \cap U_\beta$$

$\therefore$　f は global な const fn

$\therefore$　M conn $\Leftrightarrow$ $H^0(M.\mathbb{Z}) \cong \mathbb{Z}$

<u>Ex</u>　$[f] \in \check{H}^1(M.\mathbb{Z})$

$\mathcal{U} = \{U_\alpha\}$ s.t.　$f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{Z}$　locally const

s.t.　$(\delta f)_{\alpha\beta\gamma} = f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta} = 0$

on $U_{\alpha\beta\gamma}$

$f = \{f_{\alpha\beta}\} \sim g = \{g_{\alpha\beta}\}$

$\Leftrightarrow \quad {}^{\exists} h = \{h_\alpha\} \in C^0(\mathcal{U}, \mathbb{Z})$ s.t.

$$g_{\alpha\beta} - f_{\alpha\beta} = (\delta h)_{\alpha\beta} = h_\beta - h_\alpha$$

$$(\text{i.e.} \quad g_{\alpha\beta} = -h_\alpha + f_{\alpha\beta} + h_\beta \ )$$

c.f. $\quad H^1(M, \underline{G})$ では $\quad g_{\alpha\beta} = h_\alpha^{-1} f_{\alpha\beta} h_\beta$

<u>Fact</u>

$$\check{H}^p(M, \mathbb{Z}) \cong H^p(M, \mathbb{Z})$$

usual cohomology

<u>orientation & 1st Stiefel Whitney class</u>

$E \to M$ rank m real vector bundle

with fiber metric h

$\rightsquigarrow \quad O(E)$ E の o.n.f bundle

tran fn $\quad g_{\alpha\beta} : U_\alpha \cap U_\beta \to O(m)$

$$[\{g_{\alpha\beta}\}] \in H^1(M, \underline{O(m)})$$

$$z_{\alpha\beta} = \det(g_{\alpha\beta}) : U_{\alpha\beta} \to \mathbb{Z}_2 = \{\pm 1\}$$

$$w_1(E) = [\{z_{\alpha\beta}\}] \in H^1(M, \mathbb{Z}_2)$$

1-st Stiefel-Whitney class

$w_1(M) = w_1(TM)$

$\because$) $\delta g = I$ より $\delta z = 1 \in \mathbb{Z}_2$

$\therefore$ $[\{z_{\alpha\beta}\}] \in H^1(M.\ \mathbb{Z}_2)$

また、$g'_{\alpha\beta} = f_\alpha^{-1} g_{\alpha\beta} f_\beta$ のとき、$\tau_\alpha = \det f_\alpha$

とすると $z'_{\alpha\beta} = \tau_\alpha^{-1} \times z_{\alpha\beta} \times \tau_\beta = z_{\alpha\beta} \times (\delta\tau)_{\alpha\beta}$

in $\mathbb{Z}_2$

$\therefore$ $[\{z_{\alpha\beta}\}] = [\{z'_{\alpha\beta}\}]$ //

また $w_1(\cancel{M}\ E)$ trivial

$\Longleftrightarrow$ E is orientable

$\because$) $\{z_{\alpha\beta}\}$ is trivial

$z_{\alpha\beta} = (\delta\tau)_{\alpha\beta}$ $\quad {}^\exists \tau_\alpha \in C^0(\mathcal{U}.\ \mathbb{Z}_2)$

$\tau_\alpha = 1 \Rightarrow f_\alpha = I$

$\in O(n)$

$\tau_\alpha = -1 \Rightarrow f_\alpha = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

$g'_{\alpha\beta} = f_\alpha^{-1} g_{\alpha\beta} f_\beta$

$\det g'_{\alpha\beta} = z_{\alpha\beta} \cdot (\delta\tau)_{\alpha\beta} = 1$

$\therefore$ E is orientable, 逆は easy //

cpx line bundle & 1-st Chern class

$L \to M$ cpx line bundle

$g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow GL(1;\mathbb{C}) = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$

必要なら細分をとり $U_{\alpha\beta}$ simply conn

$g_{\alpha\beta} = \exp 2\pi i \ {}^{\exists}k_{\alpha\beta} \qquad k_{\alpha\beta} : V_{\alpha\beta} \to \mathbb{C}$

$C_{\alpha\beta\gamma} := k_{\alpha\beta} + k_{\beta\gamma} + k_{\gamma\alpha} : U_{\alpha\beta\gamma} \to \mathbb{Z}$

locally const

$(\because \ g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = I)$

$c_1(L) := -[\{C_{\alpha\beta\gamma}\}] \in H^2(M.\mathbb{Z})$

L の 1-st chern class ${}_{//}$

Spin str and 2-nd Stiefel Whitney class

(M^n, g) oriented Riem mfd

$SO(M) \quad$ tran fn $\quad h_{\alpha\beta} : U_{\alpha\beta} \to SO(n)$

$\downarrow \pi$

$M \qquad\qquad U_{\alpha\beta}$ simply conn

$\exists \ \tilde{h}_{\alpha\beta} : U_{\alpha\beta} \to Spin(n) \quad$ s.t. $Ad\,\tilde{h} = h$

このとき

$z_{\alpha\beta\gamma} := \tilde{h}_{\alpha\beta} \tilde{h}_{\beta\gamma} \tilde{h}_{\gamma\alpha} \in \mathbb{Z}_2$

$\because)$

$$\mathrm{Ad}(\tilde{h}_{\alpha\beta}\tilde{h}_{\beta\gamma}\tilde{h}_{\gamma\alpha}) = h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha} = I$$

$\therefore \quad z_{\alpha\beta\gamma} \in \ker \mathrm{Ad} = \mathbb{Z}_2$

$(\delta z)_{\alpha\beta\gamma\nu} = \cdots\cdots = 1$

$\therefore \quad w_2(M) := [\{z_{\alpha\beta\gamma}\}] \in H^2(M, \mathbb{Z}_2)$

2-nd Stiefel Whitney class.

<u>Exc</u> $(\delta z)_{\alpha\beta\gamma\nu} = 1$ を示せ

<u>Exc</u> $h_{\alpha\beta} \rightsquigarrow \tilde{h}_{\alpha\beta}$ lift

$\rightsquigarrow \tilde{h}'_{\alpha\beta} = \tilde{h}_{\alpha\beta} \cdot {}^{\exists}\omega_{\alpha\beta}$ 別のlift

$\omega_{\alpha\beta} \in C^1(\mathcal{U}, \mathbb{Z}_2)$

このとき $[\{z_{\alpha\beta\gamma}\}] = [\{z'_{\alpha\beta\gamma}\}]$ を示せ。

さて $\{\tilde{h}_{\alpha\beta}\}$, $\tilde{h}_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Spin}(n)$

が cocycle condition をみたせば

(i.e. $z_{\alpha\beta\gamma} = \tilde{h}_{\alpha\beta}\tilde{h}_{\beta\gamma}\tilde{h}_{\gamma\alpha} = I \quad \forall \alpha, \beta, \gamma$)

pr Spin(n)-bundle on M を得る

<u>**Def**</u> (M^n, g) oriented Riem mfd.

pr $Spin(n)$-bundle $Spin(M)$

$\Phi : Spin(M) \to SO(M)$ bundle homo

s.t.

$$\begin{tikzcd}
Spin(M)\times Spin(n) \arrow[rr, "\Phi\times Ad"] \arrow[d] \arrow[drr, phantom, "\circlearrowright"] & & SO(M)\times SO(n) \arrow[d] \\
Spin(M) \arrow[rr, "\Phi"] \arrow[dr, "\pi'"'] & \circlearrowright & SO(M) \arrow[dl, "\pi"] \\
& M &
\end{tikzcd}$$

組 $(Spin(M), \Phi)$ を (M, g) 上の spin structure

という.

<u>Note</u>

- $\circlearrowright$: $\Phi(ug) = \Phi(u)Ad(g)$
- $\circlearrowright$: $\pi(\Phi(u)) = \pi'(u)$
- 各 fiber 上で double covering

<u>**Def**</u> $(Spin(M), \Phi)$, $(Spin(M)', \Phi')$ が同値とは,

$\exists f : Spin(M) \to Spin(M)'$ isom as pr $Spin(n)$ bdl

s.t. $\underline{\Phi' \circ f = \Phi}$

f: bdle isom としても,

これを満たさないなら spin str として同値とは言わない。

Thm (M, g) oriented Riem mfd

(M, g) が spin str をもつ $\Leftrightarrow$ $w_2(M)$ trivial

spin str の同値類 $\cong H^1(M, \mathbb{Z}_2)$

(ある spin str を fix したら)

proof $\Rightarrow$ $\tilde{h}_{\alpha\beta}$ s.t. $\delta\tilde{h} = I$ となる

tran fn of Spin(M) がとれる

$\therefore$ $w_2(M) = 1$

$\Leftarrow$ $w_2(M) = [\{z_{\alpha\beta\gamma}\}] = 1$ とすると

$z_{\alpha\beta\gamma} = (\delta\nu)_{\alpha\beta\gamma} = \nu_{\beta\gamma}\nu_{\alpha\gamma}^{-1}\nu_{\alpha\beta}$

となる $\{\nu_{\alpha\beta}\} \in C^1(\mathcal{U}, \mathbb{Z}_2)$ を得る

$\hat{h}_{\alpha\beta} := \nu_{\alpha\beta}\tilde{h}_{\alpha\beta}$ とすると $\delta\hat{h} = 1$

$\therefore$ $\{\hat{h}_{\alpha\beta}\}$ は spin str を与える. //

$\{\hat{h}_{\alpha\beta}\}$, $\{\tilde{h}_{\alpha\beta}\}$ spin str とする

$Ad(\hat{h}_{\alpha\beta}) = Ad(\tilde{h}_{\alpha\beta})$ より

$\{\tau_{\alpha\beta}\} \in C^1(\mathcal{U}, \mathbb{Z}_2)$ s.t. $\hat{h}_{\alpha\beta} = \tau_{\alpha\beta}\tilde{h}_{\alpha\beta}$

$\delta\hat{h} = I$, $\delta\tilde{h} = I$ より $\delta\tau = 1$

$\therefore$ $[\{\tau_{\alpha\beta}\}] \in H^1(M, \mathbb{Z}_2)$

もし, $[\{\tau_{\alpha\beta}\}] = 1$ なら $\tau_{\alpha\beta} = \omega_\alpha^{-1}\omega_\beta$ となる

$\{w_\alpha\} \in C^0(\mathcal{U}, \mathbb{Z}_2)$ を得る

$\hat{h}_{\alpha\beta} = \tau_{\alpha\beta}\tilde{h}_{\alpha\beta} = w_\alpha^{-1}\tilde{h}_{\alpha\beta}w_\beta$

$\therefore$ $\{\hat{h}_{\alpha\beta}\}$, $\{\tilde{h}_{\alpha\beta}\}$ は同値な spin str //

Note

$$H^0(M, \underline{SO(n)}) \xrightarrow{\delta^*} H^1(M, \mathbb{Z}_2) \xrightarrow{i} H^1(M, \underline{Spin(n)})$$

$$\xrightarrow{Ad} H^1(M, \underline{SO(n)}) \xrightarrow{\delta^*} H^2(M, \mathbb{Z}_2) \quad \text{exact seq}$$

$$\begin{array}{ccc} \overset{\psi}{[h]} & \longmapsto & w_2(M) \end{array}$$

$\tau \in \delta^*(H^0(M, \underline{SO(n)})) \subset H^1(M, \mathbb{Z}_2)$

$\hat{h}_{\alpha\beta} = \tau_{\alpha\beta}\tilde{h}_{\alpha\beta}$

このとき $\hat{h}$ と $\tilde{h}$ は pr Spin(n) bundle

として isom. しかし, spin str としては

同値でない //

Def

(M, g) oriented Riem + spin str $Spin(M)$

を (Riem) spin mfd という //

Ex

$S^n\ (n \geq 3)$ $\quad H^1(M, \mathbb{Z}_2) = 0$

$H^2(M, \mathbb{Z}_2) = 0$

$\therefore\ w_1(M), w_2(M)$ trivial

S^n は unique spin str をもつ

$n=2$ $\quad w_2(S^2)$ is trivial. S^2 も unique spin str

$n=1$

$M = S^1$ $\quad SO(1) = \{1\}$, $Spin(2) = \{\pm 1\}$

$SO(S^1) \cong S^1$, $\quad H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2$

$\rightarrow$ 2つの spin str をもつ

Stiefel Whitney class の計算 (Milnor など)

により

$\mathbb{R}P^n$ spin $\Leftrightarrow$ $n \equiv 3 \mod 4$

$\mathbb{C}P^n$ spin $\Leftrightarrow$ n : odd

・ー ー・・ "

Spinor bundle

(M^n, g) spin mfd

$\mathrm{Spin}(M)$ pr $\mathrm{Spin}(n)$-bundle

$\Delta_n : \mathrm{Spin}(n) \curvearrowright W_n$ spinor rep

Def

$$S := \mathrm{Spin}(M) \times_{\Delta_n} W_n \quad \text{cpx vector bundle}$$

を spinor bundle という

$s \in \Gamma(M, S)$ を spinor field という.

locally

$s_\alpha : U_\alpha \to W_n$ s.t. $s_\alpha = \Delta_n(\tilde{h}_{\alpha\beta}) s_\beta$ 〃

- $\langle\, , \,\rangle$: $\mathrm{Spin}(n)$-inv Herm inner pr on W_n

  $\rightsquigarrow$ Herm fiber metric on S

- n = even $\quad S = S^+ \oplus S^-$

  $$S^\pm = \mathrm{Spin}(M) \times_{\Delta^\pm} W_n^\pm$$

  $\omega_{\mathbb{C}}$ acts by ± 1

- $Cl_n \times W_n \to W_n$ Clifford multi

  は $\mathrm{Spin}(n)$-eq

$$\Gamma(Cl(M)) \times \Gamma(S) \ni (\phi, \psi) \longmapsto \phi \cdot \psi \in \Gamma(S)$$

$z < 1 =$ $\quad X \in \mathfrak{X}(M)$

$$X \cdot (S^{\pm}) \subset S^{\mp}$$

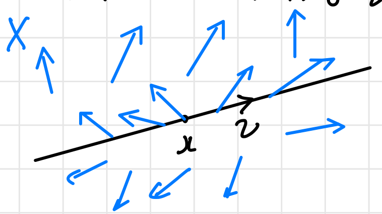
§ 接続と共変微分

connection and covariant derivative

$\mathbb{R}^n \quad X = (X_1(x), \cdots, X_n(x))$ vector field

$x \in \mathbb{R}^n, \quad v \in T_x\mathbb{R}^n,$

Xの点 xにおける v方向への方向微分とは,



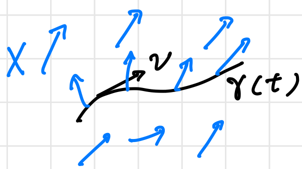
$$\nabla_v X = \lim_{t\to 0} \frac{X(x+vt) - X(x)}{t}$$

$$X(x+vt) \in T_{x+vt}\mathbb{R}^n \cong \mathbb{R}^n$$

$$X(x) \in T_x\mathbb{R}^n \cong \mathbb{R}^n$$

自然な同一視

M manifold, $X \in \mathfrak{X}(M) = \Gamma(M, TM)$



$\gamma'(0) = v \in T_xM$

$T_xM \sim T_{\gamma(t)}M$

??

同一視しないと $X(\gamma(t)) - X(\gamma(0))$ が考えられない

$\rightsquigarrow$ 接続 or 平行移動 (or 共変微分)

$$\begin{cases} \mathbb{R}^n & \dfrac{\partial^2 X}{\partial u \partial v} - \dfrac{\partial^2 X}{\partial v \partial u} = 0 \\ M & \nabla_u \nabla_v X - \nabla_v \nabla_u X = \underbrace{R(u,v)}_{\text{曲率}} X \neq 0 \end{cases}$$

Connection on P

G Lie group $\qquad L_g : G \ni x \mapsto gx \in G$
$\qquad\qquad\qquad\quad R_g \qquad\qquad\qquad\; xg$

- $\mathfrak{g} = \mathrm{Lie}(G)$

  $= \{ X \in \mathfrak{X}(M) \mid dL_g(X) = X, \ \forall g \in G \}$

  Lie subalg of $\mathfrak{X}(G)$

  $\cong T_e G$

- $X \in \mathfrak{g} \rightsquigarrow$ integral curve s.t. $\gamma(0) = e$

  $\exp tX \qquad$ 1 parameter subgr

flow of X は $R_{\exp tX}(g) = g(\exp tX)$

- $L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g : G \to G$

$\rightsquigarrow \ Ad(g) := d(L_g \circ R_{g^{-1}}) : \mathfrak{X}(G) \to \mathfrak{X}(G)$

$\rightsquigarrow \ Ad : G \ni g \mapsto Ad(g) \in GL(\mathfrak{g})$ gr homo

& $Ad(g)[X, Y] = [Ad(g)X, Ad(g)Y]$

$\rightsquigarrow \ ad := (dAd)_e : T_eG = \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$

は $ad(X)(Y) = [X, Y]$ であり

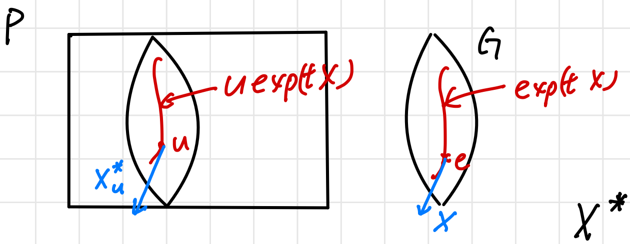
$$\begin{tikzcd} G \arrow[r, "Ad"] & GL(\mathfrak{g}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "ad"'] \arrow[ur, phantom, "\circlearrowright"] & \mathfrak{gl}(\mathfrak{g}) \arrow[u, "\exp"'] \end{tikzcd}$$

さて $\begin{array}{c} P \curvearrowleft G \\ \downarrow \\ M \end{array}$ pr G-bundle

$R_g : P \ni u \mapsto ug \in P$

$X \in \mathfrak{g}$ に対して $X^* \in \mathfrak{X}(P)$ を

$$(X^*)_u := \frac{d}{dt}(u \exp tX)\Big|_{t=0}$$ とする



X^* : fundamental vector field

X^* の flow on P $= R_{\exp tX}$

Lem $\quad dR_g(X^*) = (Ad(g^{-1})X)^*$

proof

$$(dR_g X^*)_u = (dR_g)_{ug^{-1}} X^*_{ug^{-1}} = \frac{d}{dt} R_g R_{\exp tX}(ug^{-1})\Big|_{t=0}$$

$$= \frac{d}{dt} u \underbrace{g^{-1}(\exp tX)g}_{= \exp(t Ad(g^{-1})X)} = (Ad(g^{-1})X)^*_u$$

prop

$\mathfrak{g} \ni X \mapsto X^* \in \mathfrak{X}(P)$ Lie alg homo proof

$$[X^*, Y^*] = \frac{d}{dt} dR_{\exp(-tX)}(Y^*)\Big|_{t=0}$$

Lem

$$= \frac{d}{dt}\Big(\underbrace{Ad(\exp tX)}_{= \exp t\, ad(X)} Y\Big)^*\Big|_{t=0} = [X, Y]^* \quad //$$

$$\begin{array}{c} P \curvearrowleft G \\ \pi \downarrow \\ M \end{array} \qquad u \in P$$

$$V_u = \{ w \in T_uP \mid d\pi(w) = 0 \}$$

$$V := \bigcup_{u \in P} V_u \subset TP \quad \text{sub bundle}$$

$$\downarrow$$

$$P$$

V : vertical bundle

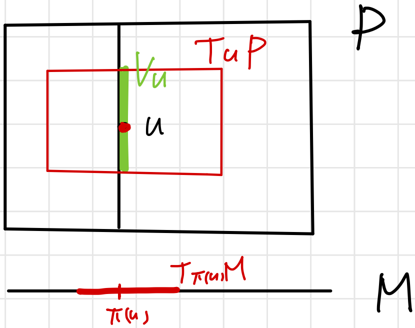
- $V_u = \{ X_u^* \mid X \in \mathfrak{g} \} \cong \mathfrak{g}$
- $dR_g(V) = V \quad (dR_g(V_u) = V_{ug})$

∵) by the previous Lem

一方, $\pi^* TM \to P$ vector bundle

$(\pi^* TM)_u \cong T_{\pi(u)} M$ であり,

$T_uP / V_u \cong T_{\pi(u)} M$



G-inv ($dR_g(V)=V$)

$$0 \longrightarrow V \xrightarrow{i} TP \xrightarrow{d\pi} \pi^*TM \to 0$$

P上 vector bundle の exact seq

(i.e. 各 fiber で exact seq)

$$0 \longrightarrow V \xrightarrow{i} TP \underset{\longleftarrow}{\xrightarrow{d\pi}} \pi^*TM \to 0$$

j s.t $d\pi \circ j = id$

G inv-map で $TP = V \oplus j(\pi^*TM)$

を与えることを "接続" という

(そのような j はいろいろ. つまり接続の与え方は、色々ある。)

<u>Def</u>

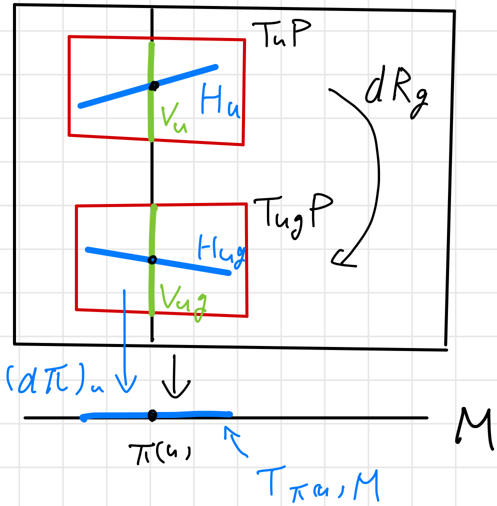
$P \circlearrowleft G$ 主G束
$\downarrow$
M

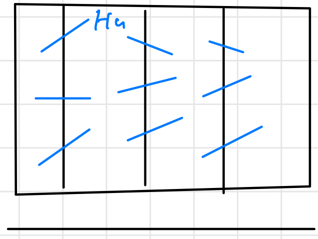
P上の connection とは,

TP の sub bundle H で

- $TP = V \oplus H$ ($\therefore$ $H \cong \pi^*TM$)
- $dR_g(H_u) = H_{ug}$ ($\forall u \in P,\ g \in G$)

をみたすもの。 (H は horizontal bundle ともいう)





$$H = \bigcup_{u \in P} H_u$$

Conn H に associate する conn 1-form をつくる

$\Omega^1(P) \otimes \mathfrak{g} = \{ P$ 上 $\mathfrak{g}$-valued 1-forms on $P \}$

$A = \sum^n A_i \otimes X_i$ $\quad$ ($\{X_i\}$ basis of $\mathfrak{g}$)

$G \curvearrowright \Omega^1(P) \otimes \mathfrak{g}$ by

pull back by R_g

$$g \cdot A := \sum_{i=1}^{n} R_g^*(A_i) \otimes Ad(g) X_i$$

Given conn H on P

$A \in (\Omega^1(P) \otimes \mathfrak{g})^G$ を

$$A(v) := \begin{cases} 0 & v \in H_u \subset T_uP \\ X & v = X_u^* \quad (X \in \mathfrak{g}) \end{cases}$$

Note $\quad v \in V_u \Rightarrow \exists ! X \in \mathfrak{g}$ s.t. $v = X_u^*$

<u>proof</u> A : G-inv を示す.

$v \in H_u$ のとき $\qquad\qquad\qquad\qquad dR_g(H_u) = H_{ug}$

$$(g \cdot A)(v) = \sum A_i(dR_g(v)) Ad(g) X_i = 0$$

$v = X_u^*$ のとき

$$(g \cdot A)(X_u^*) = \sum A_i(\overbrace{dR_g(X_u^*)}^{= (Ad(g^{-1})X)_u^*}) Ad(g) X_i$$

$$= \sum A_i(X_u^*) X_i = X \quad //$$

<u>Def</u>

A が P 上 connection 1-form とは,

(1) $\quad A \in (\Omega^1(P) \otimes \mathfrak{g})^G \leftarrow$ G-inv

(2) $\quad A$ is vertical i.e. $A(X^*) = X \; ({}^\forall X \in \mathfrak{g})$

A as above

$$\begin{array}{l} H := \bigcup_{u \in P} \ker A_u \\ \downarrow \\ P \end{array} \qquad \ker A_u = \{ v \in T_uP \mid A_u(v) = 0 \}$$

は G-inv horizontal bundle

つまり conn on P

$\because)$ $\quad A(X^*) = X$ より, $\ker A_u \oplus V_u = T_uP$

A : G-inv より, $\qquad dR(H) = H \quad //$

Ex

A_{MC} : G上 Maurer-Cartan 1-form

$\in \Omega^1(G)\otimes\mathfrak{g}$ def by

$$(A_{MC})_g(X_g) := X \qquad ({}^\forall X\in\mathfrak{g})$$

$P = M\times G$ trivial pr G-bundle

$$\begin{tikzcd} P \arrow[d, "\pi"] \arrow[r, "\pi_G"] & G \\ M & \end{tikzcd}$$

$A^0 := \pi_G^* A_{MC}$ は conn 1-form on P

$$\Big(T_uP = \underset{\substack{\shortparallel \\ H_u}}{T_xM} \oplus \underset{\substack{\shortparallel \\ V_u}}{T_gG} \qquad u=(x,g) \Big)$$

Ex

$$\begin{tikzcd} P \arrow[d] & \arrow[l, bend right] G \\ M & \end{tikzcd}$$

$\pi^{-1}(U_\alpha)\cong U_\alpha\times G$

A^0_α : $\pi^{-1}(U_\alpha)$ 上 trivial conn

$M = \bigcup_{\alpha\in A} U_\alpha$ $\{\rho_\alpha\}$ 1の分割

$P = \bigcup_{\alpha\in A}\pi^{-1}(U_\alpha)$

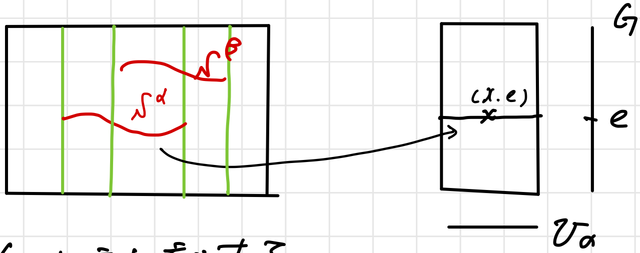
$A = \sum \pi^*\rho_\alpha A^0_\alpha$ は P上 conn 1-form

A の local 表示

$P \curvearrowleft G$ $\qquad \pi^{-1}(U_\alpha) \simeq U_\alpha \times G$

$\downarrow$ $\qquad$ G-eq trivalization

$M = \bigcup_{\alpha \in A} U_\alpha$



$(x, e) \in U_\alpha \times G$ に対応する

local section を $s^\alpha \in \Gamma(U_\alpha, P)$ とする(各 α)

このとき $\quad s^\beta = s^\alpha g_{\alpha\beta}$ となる

(G は右から作用)

<u>Def</u> $\quad A$ conn 1-form

$A_\alpha := (s^\alpha)^*(A)$ $\quad U_\alpha$ 上 $\mathfrak{g}$-valued 1-form

$\{A_\alpha\}_{\alpha \in A}$ の はり合わせは

$$A_\beta = Ad(g_{\alpha\beta}^{-1}) A_\alpha + g_{\alpha\beta}^{-1} d g_{\alpha\beta} \quad *$$

on $U_\alpha \cap U_\beta$

"local gauge transformation"

$\{A_\alpha\}$ s.t $*$ から conn 1-form A は再現できる

$\because$) U_α 上 1-form A_α を G 作用で $\pi^{-1}(U_\alpha)$ へ

proof of ✱

まず $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$

$(dg_{\alpha\beta})_x : T_xM \to T_{g_{\alpha\beta}(x)}G$

$g_{\alpha\beta}(x)^{-1}(dg_{\alpha\beta}) : T_xM \to T_eG = \mathfrak{g}$

$\therefore$ $g^{-1}dg$ は $\mathfrak{g}$-valued 1-form

さて. $\gamma(t)$ curve in M, $\gamma(0) = x$, $\gamma'(0) = v$

$$ds^\beta(v) = \frac{d}{dt} s^\beta(\gamma(t))\Big|_{t=0} = \frac{d}{dt} s^\alpha(\gamma(t))g_{\alpha\beta}(\gamma(t))\Big|_{t=0}$$

$$= dR_{g_{\alpha\beta}(x)}ds^\alpha(v) + \underline{s^\beta(x)g_{\alpha\beta}^{-1}(x)dg_{\alpha\beta}(v)}$$

$X = g_{\alpha\beta}^{-1}(x)(dg_{\alpha\beta})_x(v) \in \mathfrak{g}$ に対する X^* の $s^\beta(x)$

における値

$$\therefore\ A_\beta(v) = (s^\beta)^*A(v) = A(ds^\beta(v))$$

$$= A(dR_g ds^\alpha(v)) + A(X^*_{s^\beta(x)})$$

$$= Ad(g_{\alpha\beta}^{-1})A_\alpha(v) + g_{\alpha\beta}^{-1}dg_{\alpha\beta}(v) \quad //$$

A is G-inv

Curvature of A

$Z, W \in \Gamma(P, H)$ のとき

$[Z, W]$ は $\Gamma(P, H)$ に入るか??

それを言えるのが curvature F_A である.

A conn 1-form on P

dA : P 上 $\mathfrak{g}$-valued 2-form

<u>Lem</u> $X, Y \in \mathfrak{g}$, $v \in H_u$

$(dA)_u(X_u^*, Y_u^*) = -[X, Y]$

$(dA)_u(X_u^*, v) = 0$

$\because$) $A(X^*) = X$ const

$$(dA)(X^*, Y^*) = \underbrace{X^*(A(Y^*))}_{=0} - \underbrace{Y^*(A(X^*))}_{=0} - A(\underset{[X.Y]^*}{\overset{}{[X^*, Y^*]}})$$

$= -[X. Y]$

$v \in H_u$ を $W \in \Gamma(P. H)$ へ extend $v = W_u$

$[X^*, W] = \frac{d}{dt} R_{\exp(-tX)}(W)\Big|_{t=0}$

$dR_g(H) = H$ より, $[X^*, W] \in \Gamma(P. H)$

$$dA(X^*, W) = X^*(\underbrace{A(W)}_{=0}) - \underbrace{W(A(X^*))}_{=0} - A([X^*, W])$$

$= 0$ //

A, B $\mathfrak{g}$-valued 1-form　　$A = \sum A_i \otimes X_i$

$$[A \wedge B] = \sum_{i,j} A_i \wedge A_j \otimes [X_i, X_j] \in \Omega^2(P) \otimes \mathfrak{g}$$

とする

Ex　$\mathfrak{g}$: abelian なら　$[A \wedge B] = 0$

Exc　$Z, W \in \mathfrak{X}(P)$ のとき

$$[A \wedge A](Z, W) = 2[A(Z), A(W)]$$ を示せ

Def

conn 1-form A の curvature とは

$$F_A := dA + \frac{1}{2}[A \wedge A]$$

Exc　F_A が G-invariant を示せ。

Lem より,

$X, Y \in \mathfrak{g}$, $Z, W \in \Gamma(P, H)$

$$F_A(X^*, Y^*) = -[X, Y] + \frac{1}{2} 2 [X, Y] = 0$$

$$F_A(X^*, W) = 0 + [A(X^*), \underbrace{A(W)}_{=0}] = 0$$

$$F_A(Z, W) = dA(Z, W) + 0$$

$$= Z(A(W)) + W(A(Z)) - A([Z, W])$$

$$= -A([Z, W])$$

∴ F_A は $[Z, W]$ の vertical 方向を言ってる。

Ex　$F_A = 0$　flat conn という

$\iff$ H は integrable distribution on P

($F_A = 0 \iff {}^\forall Z, W \in \Gamma(P, H)$, $[Z, W] \in \Gamma(P, H)$

∴ フロベニウスの定理より,

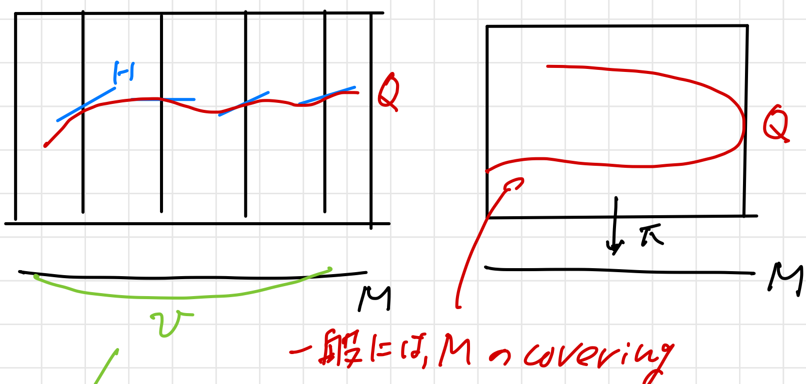
$u \in P$ を通る H の integral mfd $Q \subset P$

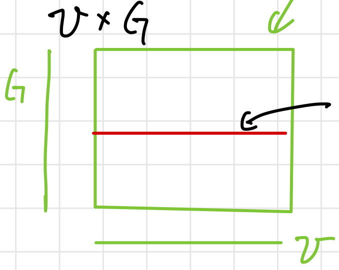
を得る　　　　　　　　　　$(T_q Q = H)$

Q は $P \to M$ の section (or M の covering)

s.t.　$P \cong M \times G$　$TM = H$

つまり A は (local) trivial conn





$e(x) = (x, e)$ が Q に対応

Exc F_A の local 表示

$$A_\alpha = (s^\alpha)^*(F_A), \quad F_\alpha := (s^\alpha)^* F_A$$

このとき

- $F_\alpha = dA_\alpha + \frac{1}{2}[A_\alpha \wedge A_\alpha]$
- $F_\beta = \mathrm{Ad}(g_{\alpha\beta}^{-1}) F_\alpha$

を示せ

$$\therefore \quad F \in \Gamma(M, \Lambda^2 T^*M \otimes P \times_{\mathrm{Ad}} \mathfrak{g}) \,//$$

parallel transport

$P \curvearrowleft G$ 　　 H conn
$\downarrow$
M 　　 $\gamma : [0.1] \to M$ piecewise smooth curve
$\gamma(0) = x$

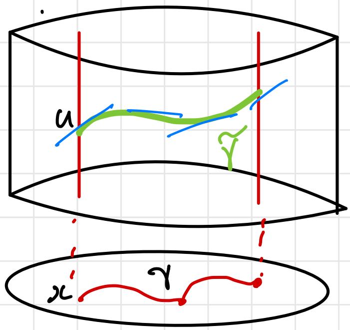
$u \in \pi^{-1}(x)$ fix

Def 　 $\tilde{\gamma}$: γ の horizontal lift とは

$\tilde{\gamma} : [0.1] \to P$. p.w smooth s.t.

$\tilde{\gamma}(0) = u$, 　　 $\pi(\tilde{\gamma}) = \gamma$,

$\gamma'(t) \in H_{\tilde{\gamma}(t)}$ 　 $(t \in I)$ //



Prop 　 $\tilde{\gamma}$ は u に対し, unique

に存在する//

∵) 　 local section s^α

$\pi^{-1}(U_\alpha) \cong U_\alpha \times G$

をとめる curve $\tilde{\gamma}$ とすると

$\tilde{\gamma}(t) = s^\alpha(\gamma(t))\, g(t) \longleftrightarrow (\gamma(t), g(t))$

$\tilde{\gamma}'(t) = dR_{g(t)}\, ds^\alpha(\gamma'(t)) + \tilde{\gamma}(t)\, g^{-1}(t)\, g'(t)$

$\therefore \quad A(\tilde{\gamma}'(t)) = Ad(g(t)^{-1})\, A_\alpha(\gamma'(t)) + g(t)^{-1} g'(t)$

よって

$\tilde{\gamma}'(t) \in H_{\tilde{\gamma}(t)}$ in $\mathfrak{g}$

$\iff \quad Ad(g(t)^{-1})A_\alpha(\gamma'(t)) + g(t)^{-1}g'(t) = 0$ ●

O.D.E の sol の exist & unique より

$\exists^1 g(t)$ s.t. ● & $u = s^\alpha(x)g(0)$

つなぎあわせれば、horizontal lift $\tilde{\gamma}$ を得る.

<u>Note</u> ug から出発したら

$\tilde{\gamma}(t)g$ が horizontal lift //

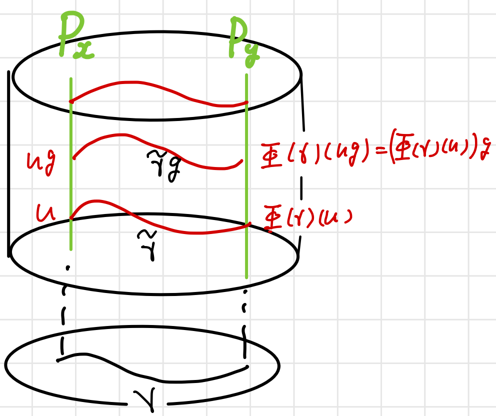
<u>Def</u> (parallel transport)

γ : curve in M, $\gamma(0) = x$, $\gamma(1) = y$ とする.

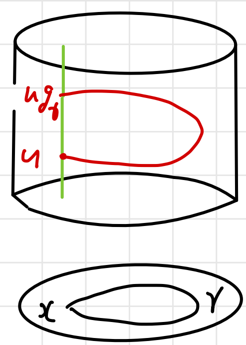
$$\begin{array}{cccc} \Phi(\gamma): & P_x = \pi^{-1}(x) & \longrightarrow & P_y = \pi^{-1}(y) \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & u = \tilde{\gamma}(0) & \longmapsto & \tilde{\gamma}(1) \end{array}$$

により、G-equivalent diffeo を得る.

γ に沿った parallel transport という.



γ : loop のとき



directional derivative

$P \curvearrowleft G$ conn H or conn 1-form A
$\downarrow$
M $\rightsquigarrow$ associated vector bundle 上へ

$$\mathbb{V} := P \times_{\rho} V$$

γ curve in M $\rightsquigarrow$ $\tilde{\gamma}$ hor lift in P
s.t. $\tilde{\gamma}(0) = u$

このとき

$$\Phi(\gamma) : \mathbb{V}_{\gamma(0)} \ni [u, v] \longmapsto [\tilde{\gamma}(1), v] \in \mathbb{V}_{\gamma(1)}$$

γに沿った parallel transport

(welldefined, $\Phi(\gamma)$ は linear isom)

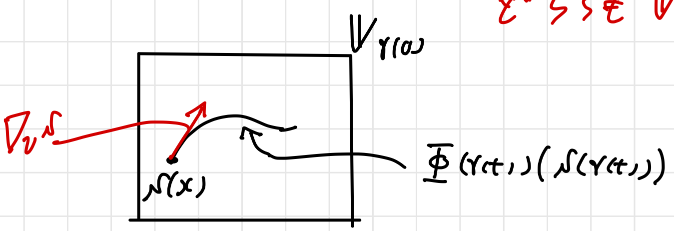
異なる点 x, y の fiber $\mathbb{V}_x, \mathbb{V}_y$ が比較できた!! $\rightsquigarrow$ 方向微分ができるはず.

<u>Def</u> $s \in \Gamma(M, \mathbb{V})$, $v \in T_xM$

γ : curve in M s.t. $\gamma(0) = x$, $\gamma'(0) = v$

$$\nabla_v s := \lim_{t \to 0} \frac{\Phi(\gamma(t))^{-1}(s(\gamma(t))) - s(x)}{t}$$

どちらも $\mathbb{V}_{\gamma(0)}$ の元



explicit formula of ∇

$P \quad A \qquad \pi^{-1}(U_\alpha) \cong U_\alpha \times G$

$\downarrow$

$M \qquad$ by $s^\alpha : U_\alpha \to P$ local section

$\mathbb{V} = P \times_\rho V \qquad V$ の basis $\mathbf{e}_1, \cdots, \mathbf{e}_r$

このとき $e_i := [s^\alpha, \mathbf{e}_i] \in \Gamma(U_\alpha, \mathbb{V})$

$e_1, \cdots, e_r$ は local frame of $\mathbb{V}$

$g(t)$ sol of

$$* \quad Ad(g(t))^{-1} A_\alpha(\gamma'(t)) + g(t)^{-1} g'(t) = 0 \text{ in } \mathfrak{g}$$

$(\gamma(t), g(t)) \in U_\alpha \times G$ が h-lift of $\gamma(t)$

とくに $g'(0) = -A_\alpha(\gamma'(0))$

また, $p^{-1}(U_\alpha) = \mathbb{V}|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times V$

$$\psi$$

$$\begin{cases} e_i(\gamma(0)) \longmapsto (x, \mathbf{e}_i) \\ \Phi(\gamma(t))^{-1}(e_i(\gamma(t))) \longmapsto (x, \rho(g(t))^{-1} \mathbf{e}_i) \end{cases}$$

ここで

$$\nabla_v e_i \mapsto \left(x, \frac{d}{dt} \rho(g(t))^{-1} \mathbf{e}_i \Big|_{t=0}\right)$$

$$= (x, -d\rho(g'(0)) \mathbf{e}_i)$$

$$\overset{*}{=} (x, d\rho(A_\alpha(v)) \mathbf{e}_i)$$

$\therefore \quad \nabla_v e_i = d\rho(A_\alpha) e_i \quad (i=1,\cdots,r)$

一般の section $s = \sum_{i=1}^{r} \xi^i e_i$ に対しては，ライプニッツ則を使えばよい

$\therefore$

$$\nabla_v s = \sum_{i=1}^{r} (v(\xi^i) + \xi^i d\rho(A_\alpha(v))) e_i$$

$$(v \in T_x M)$$

そして $X \in \mathfrak{X}(M)$

$$\nabla_X s = \sum (X(\xi^i) + \xi^i d\rho(A_\alpha(X))) e_i$$

を得る．

そして $\nabla_{fX} s = f \nabla_X s \quad ({}^\forall f \in C^\infty(M))$.

そこで $\nabla : \Gamma(M, \mathbb{V}) \to \Gamma(M, T^*M \otimes \mathbb{V})$

$$\begin{array}{ccc} \Gamma(M, \mathbb{V}) & \to & \Gamma(M, T^*M \otimes \mathbb{V}) \\ \cup\!\!\!\!\;\text{ } & & \cup \\ s & \longmapsto & \nabla s \end{array}$$

を $(\nabla s)(X) := \nabla_X s$ と定める．

このとき
- ∇ linear
- $\nabla_X(fs) = X(f)s + f\nabla_X s$

$(f \in C^\infty(M),\ X \in \mathfrak{X}(M),\ s \in \Gamma(M, \mathbb{V}))$

この ∇ を covariant derivative という

localには

$$\nabla = d + d\rho(A_\alpha) \quad \text{on } U_\alpha$$

$$\left(\begin{array}{l} \rho : G \to GL(V) \\ d\rho : \mathfrak{g} \to \mathfrak{gl}(V) \end{array} \right)$$

より一般的な def

<u>Def</u> $p: E \to M$ vector bundle

E 上の covariant derivative とは、$\Gamma(M, E)$ 上の次の

1-st order diff operator

(1) $\nabla : \Gamma(M, E) \longrightarrow \Gamma(M, T^*M \otimes E)$

(2) $\nabla(fs) = df \otimes s + f\nabla s$ ライプニッツ

$(f \in C^\infty(M),\ s \in \Gamma(M, E))$ //

γ : curve in M $\quad \gamma(0) = x,\ \gamma(0) = y$

このとき $\tilde{\gamma}$ horizontal lift とは

$\tilde{\gamma}$ curve in E s.t. $p(\tilde{\gamma}) = \gamma$, $\quad \nabla_{\gamma'} \tilde{\gamma} = 0$

locally $\nabla = d + A_\alpha$ on U_α

A_α : $\mathfrak{gl}(r;\mathbb{R})$-valued 1-form

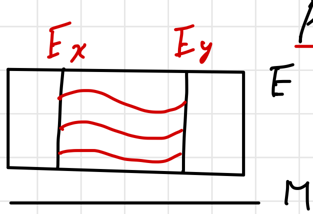
$\tilde{\gamma}(t) = (\gamma(t), v(t)) \in U_\alpha \times \mathbb{R}^r$

$v'(t) - A_\alpha(\gamma'(t))\underset{v(t)}{\wedge} = 0$ linear O.D.E

$\therefore$

$\Phi(\gamma) : E_x \ni v \longmapsto \tilde{\gamma}(1) \in E_y$ linear isom

parallel transport w.r.t ∇



Def $p : E \longrightarrow M$ with ∇

$s \in \Gamma(M, E)$ が parallel section とは

$\nabla s = 0$ となること//

次は明らか

- $\nabla s = 0 \iff {}^\forall \gamma$ curve, $\Phi(\gamma)(s) = s$
- M conn とする。$\nabla s = 0$

  $\Rightarrow$ s は 1点 x_0 での値で決まる

Curvature of ∇

$p: E \to M$ with ∇

<u>Def</u> $X, Y \in \mathfrak{X}(M)$

$$R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

<u>Exc</u> $R(fX, Y)s = R(X, fY)s = R(X,Y)(fs)$
$= f R(X,Y)s$

$\left({}^\forall X, Y \in \mathfrak{X}(M), \ {}^\forall s \in \Gamma(M, E) \ {}^\forall f \in C^\infty(M) \right)$

を示せ

- $R(X,Y) = -R(Y,X)$ を示せ

この Exc より、 $R \in \Gamma(M, \Lambda^2 T^* M \otimes \mathrm{End}(E))$

$$\begin{array}{l} P \curvearrowleft G \text{ with } A \rightsquigarrow F_A \\ \pi \downarrow \\ M \end{array}$$

$$\begin{array}{l} \mathbb{V} = P \times_\rho V \\ p \downarrow \\ M \end{array} \qquad \nabla = d + d\rho(A_\alpha) \qquad A_\alpha = s^{\alpha *} A$$

$R = ??$

<u>prop</u>

R is locally $d\rho(s^{\alpha *} F_A)$

$(\ d\rho : \mathfrak{g} \to \mathrm{End}(V) \ )$

$\underline{\text{proof}}$ $\quad \nabla e_i = d\rho(A_\alpha)e_i = \sum \omega^i{}_j \otimes e_j$

$(\omega^i{}_j)_{i,j}$ $\;U_\alpha$上 $\mathrm{End}(V)$値 1-form

$$\nabla_X \nabla_Y e_i = \sum X(\omega^i{}_j(Y))e_j + \sum \omega^i{}_j(Y)\omega^j{}_k(X)e_k$$

$$\therefore \quad R(X,Y)e_i = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})e_i$$

$$= \underline{\sum X(\omega^i{}_j(Y)) - Y(\omega^i{}_j(X))e_j}$$

$$+ \sum (\omega^i{}_j(Y)\omega^j{}_k(X) - \omega^i{}_j(X)\omega^j{}_k(Y))e_k$$

$$\underline{- \sum \omega^i{}_j([X,Y])e_j}$$

$$= d\rho\left(dA_\alpha + \frac{1}{2}[A_\alpha \wedge A_\alpha]\right)(X,Y)e_i$$

$$= d\rho(s^{\alpha *}F_A)(X,Y)e_i \;//$$

$$G \circlearrowright V$$

$$\begin{array}{ccccc} P \curvearrowleft G & A \text{ conn} & \rightsquigarrow & \mathbb{V} = P \times_\rho V & \nabla \\ \downarrow & & & \downarrow & \\ M & & & M & \end{array}$$

$v_0 \in V$ $\;G$-inv vector

$s(x) = [u, v_0]$ $\;$ section of $\mathbb{V}$

& $\;\nabla s = 0$ $\;$ (i.e. parallel)

($\because$ $\nabla_v s$ の def より)

§ Review of Riemannian geometry

(M, g) Riem mfd

∇ covariant derivative on TM

$\rightsquigarrow$ ∇ 〃 on $T^{(r,s)}M = \overset{r}{\otimes} TM \otimes \overset{s}{\otimes} T^*M$

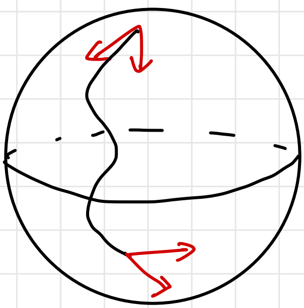
by $\cdot\ \nabla_Y(X_1 \otimes X_2) := \nabla_Y X_1 \otimes X_2 + X_1 \otimes \nabla_Y X_2$

$\cdot\ Y(\varphi(X)) =: (\nabla_Y \varphi)(X) + \varphi(\nabla_Y X)$

Ex $g \in \Gamma(M, T^{(0,2)}M)$

$\nabla g = 0 \iff \nabla_X g = 0 \quad (\forall X)$

$\iff X(g(Y, Z)) = \cancel{(\nabla_X g)(Y, Z)} + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$



$\nabla g = 0$

parallel transport で

長さ, 角度は保たれる

(∇ への自然な条件)

Ex $f \in C^\infty(M)$

$\nabla df \in \Gamma(M, T^{(0,2)}M)$

これは

$(\nabla_X df)(Y) = (\nabla_Y df)(X) \quad {}^\forall f \in C^\infty(M),\ {}^\forall X, Y \in \mathfrak{X}(M)$

$\iff T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0$

Def (M, g) Riem mfd

∇ : Levi Cita conn とは

∇ covariant derivative on TM s.t.

$$\nabla g = 0, \quad T = 0$$

Thm

(M, g) 上 L-C-conn は unique に存在

具体的には koszul formula

$$X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y))$$

$$= 2g(\nabla_X Y, Z)$$

$$+ g(Y, [X, Z]) + g([Y, Z], X) - g(Z, [X, Y])$$

Def ∇ L-C connection

∇ の curvature R を Riem curv tensor という

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

(1.3) tensor

or $\quad R(X, Y, Z, W) := g(R(X, Y)Z, W)$

Prop R は次をみたす.

(1) $R(X,Y) = -R(Y,X)$

(2) $R(X,Y,Z,W) = -R(X,Y,W,Z)$

(3) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

(第1ビアンキ恒等式)

(4) $R(X,Y,Z,W) = R(Z,W,X,Y)$

(5) $(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0$

(第2ビアンキ恒等式)

Note (1)〜(4) より

$$R \in \Gamma(M, S^2(\Lambda^2(M)))$$

symm tensor

local 表示

local coordinate $(x_1, \cdots, x_n)$

$\partial_i = \frac{\partial}{\partial x_i}$ $i = 1, \cdots, n$ とすると

$\nabla_{\partial_i} \partial_j = \sum_{l=1}^{n} \Gamma^l_{ij} \partial_l$ と書くと. koszul 公式より.

$$\Gamma^l_{ij} = \frac{1}{2} \sum_k g^{kl} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$$

Ex

$X = \sum X^i \partial_i \in \mathfrak{X}(M)$ に対して $\quad \sum \Gamma_{kj}^i \partial_i$

$$\nabla_{\partial_k} X = \sum_i \frac{\partial X^i}{\partial x_k} \partial_i + \sum_j X^j \nabla_{\partial_k} \partial_j$$

$$= \sum_i \underbrace{\left(\frac{\partial X^i}{\partial x_k} + \sum_j \Gamma_{kj}^i X^j \right)}_{= \nabla_k X^i \text{ と書く}} \partial_i$$

Exc $\quad \alpha = \sum \alpha_i dx_i \in \Omega^1(M)$ に対して

$\nabla_{\partial_k} \alpha = \sum (\nabla_k \alpha_i) dx_i$ とすると

$\nabla_k \alpha_i = \partial_k \alpha_i - \sum_j \alpha_j \Gamma_{ki}^j$ を示せ〃

Def

$$R(\partial_i, \partial_j)\partial_k = \sum R_{ijk}{}^l \partial_l$$

$$g(R(\partial_i, \partial_j)\partial_k, \partial_l) = R_{ijkl} \left(= \sum R_{ijk}{}^s g_{sl} \right)$$

により $R_{ijk}{}^l$, R_{ijkl} を def する.

prop より.

$$R_{ijkl} = -R_{jikl} = -R_{ijlk}$$

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0$$

$$R_{ijkl} = R_{klij}$$

$$\nabla_p R_{ijkl} + \nabla_i R_{jpkl} + \nabla_j R_{pikl} = 0$$

Def

$$R_{ij} := \sum R_{kij}{}^{k} \quad (= R_{ji}) \qquad \text{or}$$

$$Ric(X.Y) = \sum R_{ij} X^i Y^j \qquad X = \sum X^i \partial_i$$

によるで (0.2) tensor を Ricci curvature という

Def

$$Scal = \sum_{i.j} g^{ij} R_{ij} \quad \in C^\infty(M)$$

を Scalar curvature という

Def

$$E_{ij} = R_{ij} - \frac{Scal}{n} g_{ij}$$

を traceless Ricci Tensor という

Note $\quad tr(E) = \sum g^{ij} E_{ij} = 0$

Def

(1) $Ric = \lambda g$ ($^\exists\lambda$ const) とする $(M.g)$ を

Einstein mfd という. $(\dim M \geqq 3)$

(2) $Ric = 0$ のとき $(M.g)$ を Ricci flat manifold

(3) $R = 0$ のとき $(M.g)$ を flat manifold

(4) $R_{ijkl} = c(g_{jk}g_{il} - g_{ik}g_{jl})$ のとき

R は constant curvature という。

Ex Calabi Yau, G_2, Spin(7).

hyper kähler は Ricci $=0$

Ex irr sym space, Nearly kähler,

Einstein-Sasaki, quaternionic kähler

は Einstein

(see "Besse")

Prop (M^n, g) conn $(n \geqq 3)$

$Ric = f g \quad (f \in C^\infty(M))$

$\Rightarrow f = const$

また, (M, g) Einstein $\Leftrightarrow E = Ric - \frac{Scal}{n} g = 0$

proof $\nabla_X R$ も prop の (1)〜(4) みたす.

また, $\nabla g = 0$ より,

Constraction $R_{ijkl} \longrightarrow \sum g^{kl} R_{kijl}$

は ∇ と可換

i.e. $(\nabla_X Ric)_{ij} = \sum g^{kl} (\nabla_X R)_{kijl}$,

$\nabla_X Scal = \sum g^{kl} (\nabla_X Ric)_{kl}$

Second Bianchi

$\nabla_p R_{ijkl} + \nabla_i R_{jpkl} + \nabla_j R_{pikl} = 0$ を contract

$$\sum g^{pk}(\nabla_p R)_{ijkl} = -(\nabla_i Ric)_{jl} + (\nabla_j Ric)_{il}$$

∴ $$\sum 2 g^{il}(\nabla_i Ric)_{lj} = \nabla_j Scal$$

$Ric_{ij} = f g_{ij}$ より、 $Scal = nf$

~~再び~~ 上式へ代入して

$$X(f) = \frac{1}{2} n X(f) \quad (\forall X \in \mathfrak{X}(M))$$

∴ $n \geqq 3$ ならば $X(f)=0$, M conn より $f = const$

とくに $E = Ric - \frac{Scal}{n} g$ なら, $Scal = const$

となり, Einstein となる。

grad, div, Δ

(M, g) Riem mfd

$$TM \underset{g}{\cong} T^*M$$

$$\mathfrak{X}(M) \cong \Omega^1(M)$$

$$X = \sum X^i \partial_i \longrightarrow \varphi_X = \sum g_{ij} X^i dx^j$$

$$X_\varphi = \sum g^{ij}\varphi_i \partial_j \longleftarrow \varphi = \sum \varphi_i dx^i$$

Def $f \in C^\infty(M)$

$$\mathrm{grad}\, f := X_{df} = \sum g^{ij} \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x_j}$$

(i.e. $g(\mathrm{grad}\, f, Y) = Y(f) \quad \forall Y$)

Def $\varphi = \sum \varphi_i dx^i \in \Omega^1(M)$

$$\operatorname{div} \varphi := \sum g^{ij} \nabla_j \varphi_i \quad \left(= \sum (\nabla_{e_i} \varphi)(e_i) \right)$$

$$\in C^\infty(M)$$

$$X = \sum X^i \partial_i \in \mathfrak{X}(M)$$

$$\operatorname{div} X := \operatorname{div} \varphi_X = \sum \nabla_i X^i$$

Def $f \in C^\infty(M)$,

$$\Delta f := -\operatorname{div} \circ \operatorname{grad}(f)$$

$$= -\sum \nabla^i (\nabla_i f) \qquad \in C^\infty(M)$$

volume element

(M, g) Riem mfd oriented

$\Lambda^n(M)$ の nowhere vanishing section として

g と向きから volume element vol_g を得る

$(\omega_1, \cdots, \omega_n)$ positive o.n.f of $TM \cong T^*M$

$$\mathrm{vol}_g := \omega_1 \wedge \cdots \wedge \omega_n$$

Exc 正の local coord $(x_1, \cdots, x_n)$

$$\Rightarrow \mathrm{vol}_g = \sqrt{\det(g_{ij})}\, dx_1 \wedge \cdots \wedge dx_n$$

Exc $X \in \mathfrak{X}(M)$

$L_X(\mathrm{vol}_g) = (\mathrm{div}\, X) \cdot \mathrm{vol}_g$ を示せ。

§ Spin connection

(M, g) oriented Riem mfd

$SO(M)=P \circlearrowleft SO(n) \qquad A_{LC} \leftarrow \nabla$: Levi Civita

$\downarrow$

M

$s^\alpha=(e_1,\cdots,e_n)$: positive o.n. frame on U_α

$A_\alpha = s^{\alpha *}A_{LC}$: U_α 上 $so(n)$-valued 1-form.

これは, $\nabla e_k = \sum g(\nabla e_k, e_j)e_j$

$= A_\alpha(e_k)$ となる

さて,

$so(n) \cong \Lambda^2(\mathbb{R}^n)$ by

$u, v \in \mathbb{R}^n$

$(u \wedge v)(x) := \langle u, x\rangle v - \langle v, x\rangle u$

そこで

$$A_\alpha = \frac{1}{2}\sum_{i,j} g(\nabla e_i, e_j) e_i \wedge e_j$$

であり $\nabla = d + A_\alpha$ on U_α

同様に R を

$$R(X,Y) = \frac{1}{2}\sum_{i,j} g(R(X,Y)e_i, e_j) e_i \wedge e_j \quad \text{on } U_\alpha$$

Lie alg of Spin(n) $(Cl_n^0 \cong k(m)$
$(k = \mathbb{R}, \mathbb{H}, \mathbb{C})$

Spin(n) は $GL(m:k)$ の Lie subgroup

∴ $\mathfrak{spin}(n)$ は $k(m)$ の Lie subalg

$a, b \in \mathfrak{spin}(n)$

$$[a, b] := ab - ba \quad \text{in } Cl_n^0$$

<u>prop</u> $\mathfrak{spin}(n) = \langle e_i e_j \mid i<j \rangle_{\mathbb{R}}$

<u>proof</u>

$$\exp(t\, e_i e_j) = \sum \frac{(t e_i e_j)^m}{m!}$$

$$= \cos t + \sin t\, e_i e_j$$

$$= -\left(\cos\tfrac{t}{2} e_i + \sin\tfrac{t}{2} e_j\right)\left(\cos\tfrac{t}{2} e_i - \sin\tfrac{t}{2} e_j\right)$$

$\in$ Spin(n) & $\gamma(0) = 1$

∴ $\gamma'(0) = e_i e_j \in \mathfrak{spin}(n)$

∴ $\langle e_i e_j \mid i<j \rangle \subset \mathfrak{spin}(n) \cong \mathfrak{so}(n)$

$\dim = \binom{n}{2}$ より $\langle e_i e_j \mid i<j \rangle_{\mathbb{R}} = \mathfrak{spin}(n)$

//

Ad : Spin(n) → SO(n) の微分は

$ad = dAd : \mathfrak{spin}(n) \to \mathfrak{so}(n)$ は

$$ad(a)x = ax - xa \qquad (a \in \mathfrak{spin}(n),\ x \in \mathbb{R}^n)$$

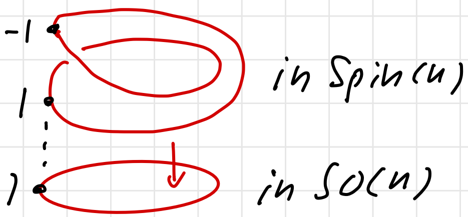
Exc

$$ad(e_i e_j) = 2\, e_i \wedge e_j$$

double cover なので

Note

$$Ad(\exp(t\, e_1 e_2)) = \left(\begin{array}{cc|c} \cos 2t & -\sin 2t & 0 \\ \sin 2t & \cos 2t & \\ \hline 0 & & I_{n-2} \end{array}\right) //$$



(M, g) spin mfd とする.

$$\begin{tikzcd} Spin(M) \arrow[rr, "\Phi"] \arrow[dr] & & SO(M) \arrow[dl] \\ & M & \end{tikzcd} \quad A,$$

$\Phi : \mathfrak{spin}(n)$-eq より. $\Phi^* A$ conn 1-form

Def

A : Levi Civita conn on $SO(M)$

$\Phi^* A$ conn on $Spin(M)$ を spin connection

という. $\mathfrak{spin}(n) \cong \mathfrak{so}(n)$-valued 1-form //

$\rightsquigarrow$ $S = Spin(M) \times_{\Delta} W_n$ 上 の ∇ を みちびく

local 表示は?

$s^{\alpha} = (e_1, \cdots, e_n)$ local section of $SO(M)$

$\tilde{s}^{\alpha}$ local section of $Spin(M)$ on U_{α}

s.t. $\Phi(\tilde{s}^{\alpha}) = s^{\alpha}$

$f_1, \cdots, f_N$ basis of W_n をとって

$\phi_k := [\tilde{s}^{\alpha}, f_k] \in \Gamma(U_{\alpha}, S)$

$\phi_1, \cdots, \phi_N$ は local frame of S

このとき

$$\nabla \phi_k = \frac{1}{4} \sum_{i,j} g(\nabla e_i, e_j)\, e_i e_j \cdot \phi_k$$

$\because)$ $d\,Ad(e_i e_j) = ad(e_i e_j) = 2\, e_i \wedge e_j$

$Spin(n) \overset{\Delta}{\curvearrowright} W_n$

$\mathfrak{spin}(n) \underset{d\Delta}{\curvearrowright} W_n$ は Clifford 積

曲率についても

$$\tilde{s}^{\alpha *} \circ \Phi^*(F_A) = (\Phi \circ \tilde{s}^{\alpha})^*(F_A) = s_{\alpha}^*(F_A)$$

より、 $R_{\Delta} = d\Delta\, s_{\alpha}^*(F_A)$ より、

prop ∇ on S, $\{e_i\}$ local p.o.n.f.

$$\nabla = d + \frac{1}{4}\sum g(\nabla e_i, e_j) e_i e_j$$

L-C

その curvature は

$$R_S(X, Y) = \frac{1}{4}\sum_{i,j} g(R(X,Y)e_i, e_j) e_i e_j$$

Riem curv

$\in End(S)$

Properties of Spin conn

$\langle\ ,\ \rangle$ Spin(n)-inv inner product on W_n

$\rightsquigarrow$ S 上に fiber metric $\langle , \rangle$ s.t.

$$\langle X\cdot\varphi, \psi\rangle + \langle\varphi, X\cdot\psi\rangle = 0 \quad {}^\forall X\in T_xM \quad {}^\forall x\in M$$
$${}^\forall\varphi, \psi \in S_x$$

これは ∇に関しても保存される

∵) ・∇ は Spin(M) の conn から induce される

・$\langle , \rangle$ は Spin(n)-inv

$\rightsquigarrow$ $\langle , \rangle$ は parallel w.r.t ∇ //

∴

$$X\langle\phi, \psi\rangle = \langle\nabla_X\phi, \psi\rangle + \langle\phi, \nabla_X\psi\rangle$$

同様に

$$\nabla_Y(X\cdot\phi) = (\nabla_Y X)\cdot\phi + X\cdot(\nabla_Y\phi)$$
$${}^\forall X, Y\in\mathfrak{X}(M) \quad {}^\forall\phi\in\Gamma(S)$$

Def $\varphi, \psi \in \Gamma(M, S)$ どちらか cpt support

$$(\varphi, \psi) := \int_M \langle \varphi, \psi \rangle \, vol_g$$

Connection Laplacian

$X, Y \in \mathfrak{X}(M)$,

$\nabla^2_{X,Y} := \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$ とする

$\Gamma(M, S)$ 上の2階微分作用素

$R_\Delta(X, Y) = \nabla^2_{X,Y} - \nabla^2_{Y,X}$ に注意.

Def

S 上の conn Laplacian を

$\nabla^* \nabla = \sum_i \nabla^2_{e_i, e_i}$ $\{e_i\}$ o.n.f

prop (M, g) cpt spin mfd

$$\int \langle \nabla^* \nabla \phi, \psi \rangle \, vol_g = \int \langle \nabla \phi, \nabla \psi \rangle \, vol_g$$

i.e. $(\nabla^* \nabla \phi, \psi) = (\nabla \phi, \nabla \psi)$

とくに $(\nabla \nabla^* \phi, \phi) = \|\nabla \phi\|^2 \geq 0$

これを示すために発散定理をもちいる

Lem (発散定理)

(M,g) cpt Riem mfd $(\partial M = \phi)$

$\Rightarrow \int_M (\mathrm{div} X) \mathrm{vol}_g = 0$

$\partial M \neq \phi$. n: 内向き単位法ベクトル

$$\int_M \mathrm{div}(X) \mathrm{vol}_g = \int_{\partial M} g(X, -n)\big|_{\partial M} \mathrm{vol}_{\partial M}$$

ここで ∂M には g からの metric を入れる.

proof $\partial M = \phi$ のとき

カルタンの公式 $L_X = d\,i(X) + i(X)\,d$ on $\Omega^p(M)$

$\therefore \mathrm{div}(X) \mathrm{vol}_g = L_X(\mathrm{vol}_g) = d\,i(X) \mathrm{vol}_g$

$\therefore \int_M \mathrm{div}(X) \mathrm{vol}_g = \int_M d(i(X)\mathrm{vol}_g) = 0$

by ストークス

$\partial M \neq \phi$ のとき

$$= \int_{\partial M} i(X) \mathrm{vol}_g\big|_{\partial M} = \int_{\partial M} g(X, -n)\big|_{\partial M} \mathrm{vol}_{\partial M}$$

//

proof of prop

まず $\nabla\phi \in \Gamma(M, S \otimes T^*M)$ であり

$S \otimes T^*M$ の fiber metric $\langle\, ,\, \rangle$ で内積をとる

つまり，$\langle \nabla\phi, \nabla\psi\rangle = \sum \langle \nabla_{e_i}\phi, \nabla_{e_i}\psi\rangle$.

さて，$g(V, W) = -\langle \nabla_W \phi, \psi\rangle$ （$\forall W \in \mathfrak{X}(M)$）により，

$V \in \mathfrak{X}(M)$ を定めると。

$$\sum_i e_i(g(V, e_i)) = -\sum_i e_i \langle \nabla_{e_i}\phi, \psi\rangle$$

$$= -\sum \langle \nabla_{e_i}\nabla_{e_i}\phi, \psi\rangle + \langle \nabla_{e_i}\phi, \nabla_{e_i}\psi\rangle$$

$$\| $$

$$\sum g(\nabla_{e_i}V, e_i) + g(V, \nabla_{e_i}e_i)$$

$$\therefore \quad \operatorname{div}(V) = \langle \nabla^*\nabla\phi, \psi\rangle - \langle \nabla\phi, \nabla\psi\rangle$$

$$\therefore \quad \int_M \langle \nabla^*\nabla\phi, \psi\rangle \mathrm{vol}_g = \int_M \langle \nabla\phi, \nabla\psi\rangle \mathrm{vol}_g$$

$$(\nabla^*\nabla\phi, \psi) \qquad\qquad (\nabla\phi, \nabla\psi) \, //$$

§ Dirac operator

$D = \sum \gamma_i \frac{\partial}{\partial x_i}$ on $C^\infty(\mathbb{R}^n, W_n)$

spin conn

$\rightsquigarrow$ $D = \sum e_i \nabla_{e_i}$ on $\Gamma(M, S)$

Clifford multi

やっと道具がそろった!!

(M, g) spin mfd, $Spin(M)$ spin str

(Δ, W_n) spinor rep

$S = Spin(M) \times_\Delta W_n$ spinor bundle

∇ spin connection

$(e_1, \cdots, e_n)$: local o.n. frame

Def

$$D := \sum e_i \cdot \nabla_{e_i} : \Gamma(M, S) \to \Gamma(M, S)$$

1st order diff op を Dirac operator という

D on $\mathbb{R}^n$ $\rightsquigarrow$ $D^2 = \Delta = \nabla^* \nabla$

D on M $\rightsquigarrow$ $D^2 = ??$

idea $\quad \nabla^2_{X,Y} - \nabla^2_{Y,X} = R_\Delta(X,Y)$ on S より

(a) $\nabla^2_{e_i,e_j} - \frac{1}{2}R_\Delta(e_i,e_j) = \nabla^2_{e_j,e_i} - \frac{1}{2}R_\Delta(e_j,e_i)$

symm w.r.t. i,j

(b) $(e_ie_j + \delta_{ij}) = -(e_je_i + \delta_{ij})$

anti symm

$\therefore \sum_{i,j} (a)(b) = -\sum_{j,i}(a)(b) = 0$

$\sum (e_i e_j + \delta_{ij})(\nabla^2_{e_i,e_j} - \frac{1}{2}R_\Delta(e_i,e_j)) = 0$

$= -\nabla^*\nabla + D^2 - \frac{1}{2}\sum e_i \cdot e_j \cdot R_\Delta(e_i,e_j) = 0$

Lem $\quad D^2 = \sum e_i \cdot e_j \cdot \nabla^2_{e_i,e_j}$

$\because)\quad D^2 = \sum (e_i\nabla_{e_i})(e_j\nabla_{e_j})$

$= \sum e_ie_j \nabla_{e_i}\nabla_{e_j} + e_i\cdot(\nabla_{e_i}e_j)\cdot\nabla_{e_j}$

$= \sum e_i e_j \nabla^2_{e_i,e_j} + e_i \cdot e_j \nabla_{\nabla_{e_i}e_j} + e_i\cdot(\nabla_{e_i}e_j)\cdot\nabla_{e_j}$

±2 $\quad \nabla_{e_i}e_j = \sum \omega(e_i)^k_j\, e_k$

$(e_1,\dots,e_n)$ o.n.f $\quad \therefore\ \omega^k_j + \omega^j_k = 0$

4 = 2.. $\quad \sum_{i,j} e_i\cdot e_j \nabla_{\nabla_{e_i}e_j} + e_i\cdot(\nabla_{e_i}e_j)\cdot\nabla_{e_j}$

$= \sum e_i\cdot e_j\, \omega(e_i)^j_k \nabla_{e_k} + e_i\cdot e_k\cdot \omega(e_i)^k_j \nabla_{e_j}$

$= 0$

$\therefore\quad D^2 = \sum e_i\cdot e_j\cdot \nabla^2_{e_i,e_j}$ //

Lem $\frac{1}{2}\sum e_i e_j R_\Delta(e_i, e_j) = \frac{1}{4}\mathrm{Scal}$

∵) まず $\sum e_i \cdot R_\Delta(X, e_i) = -\frac{1}{2}\mathrm{Ric}(X)$ を示す

(ここで $g(\mathrm{Ric}(X), Y) = \mathrm{Ric}(X, Y)$)

$R_{ijkl} = g(R(e_i, e_j)e_k, e_l)$ とすると

$$R_\Delta(X, Y) = \frac{1}{4}\sum R_{ijkl}X^i Y^j e_k e_l$$

$$\therefore\ 12\sum e_j R_\Delta(X, e_j)$$

$$= 3\sum R_{ijkl}X^i e_j e_k e_l$$

$$= \sum (R_{ijkl}X^i e_j e_k e_l + R_{iljk}X^i e_l e_j e_k + R_{iklj}X^i e_k e_l e_j)$$

$$= \sum (R_{ijkl}X^i e_j e_k e_l + R_{iljk}X^i(-2\delta_{lj}e_k + 2\delta_{lk}e_j + e_j e_k e_l) + R_{iklj}X^i(-2\delta_{lj}e_k + 2\delta_{kj}e_l + e_j e_k e_l))$$

$$\underset{\substack{\uparrow \\ \text{Bianchi}}}{=} 2\sum(-R_{illk}X^i e_k + R_{ilje}X^i e_j - R_{ikjj}X^i e_k + R_{iklk}X^i e_l$$

$$= 6\sum R_{lilk}X^i e_k = -6\,\mathrm{Ric}(X)$$

よって $\frac{1}{2}\sum e_i e_j R_\Delta(e_i, e_j) = -\frac{1}{4}\sum e_i \cdot \mathrm{Ric}(e_i)$

$$= -\frac{1}{4}\sum(\mathrm{Ric})_{il}e_i e_l = \frac{1}{4}\sum(\mathrm{Ric})_{ii} = \frac{1}{4}\mathrm{Scal}_{\,/\!/}$$

Thm (Lichnerowicz)

$$D^2 = \nabla^* \nabla + \frac{1}{4} \mathrm{Scal}$$

Cor (vanishing thm)

(M, g) cpt spin mfd

$\mathrm{Scal}(x) > 0$ $(\forall x \in M)$

or

$\mathrm{Scal}(x) \geqq 0$ $(\forall x \in M)$ & $\mathrm{Scal}(x_0) > 0$

$\Rightarrow$ $H(D) = \{0\}$

$H(D) = \{ \varphi \in \Gamma(M, S) \mid D\varphi = 0 \}$

sp of harmonic spinors (Dirac fields)

proof $D\varphi = 0$ とする

$$0 = (D^2 \varphi, \varphi) = (\nabla^* \nabla \varphi, \varphi) + \frac{1}{4} (\mathrm{Scal} \cdot \varphi, \varphi)$$

$$= \underset{\geqq 0}{(\nabla \varphi, \nabla \varphi)} + \frac{1}{4} \int_M \underset{\geqq 0}{\mathrm{Scal} |\varphi|^2} \mathrm{vol}_g \leftarrow \mathrm{Scal} \geqq 0$$

$\therefore \nabla \varphi = 0$, $\therefore \nabla \langle \varphi, \varphi \rangle = \nabla |\varphi|^2 = 0$

$|\varphi|^2 = \mathrm{const}$,

$\mathrm{Scal}(x_0) > 0$ & $0 = \int_M \mathrm{Scal} |\varphi|^2 \mathrm{vol}_g$ より

$\varphi \equiv 0$ //

properties of Dirac operator

Exc $f \in C^\infty(M)$, $\varphi \in \Gamma(M, S)$

$$D(f\varphi) = (\mathrm{grad} f)\cdot\varphi + f D\varphi$$

principal symbol

locally

$$D = \sum e_i \cdot \nabla_{e_i} = \sum g^{ij}\left(\frac{\partial}{\partial x_i}\right)\cdot\left(\frac{\partial}{\partial x_j} + A_\alpha\right)$$

for $\xi = \sum \xi_i \, dx_i \in T_x^* M \cong T_x M$,

$$\underline{\sigma_\xi(D) \text{ pr symbol}} = \sqrt{-1}\sum \xi_i g^{ij}\frac{\partial}{\partial x_i}\cdot$$

$$\underline{= \sqrt{-1}\,\xi\cdot} \quad \in \mathrm{End}(S)_x$$

とくに $\xi \neq 0$ なら $\xi\cdot\xi\cdot = -\langle\xi, \xi\rangle \mathrm{Id}$

$$(= \sigma_\xi(\Delta))$$

$\therefore$ D: elliptic operator //

formally selfadj

$\varphi, \psi \in \Gamma(S)$ どちらか cpt support

$$\langle D\varphi, \psi\rangle = \sum \langle e_i \cdot \nabla_{e_i}\varphi, \psi\rangle \cancel{=}$$

$$= -\sum \langle \nabla_{e_i}\varphi, e_i\cdot\psi\rangle$$

$$= -\sum\left(e_i\langle\varphi, e_i\psi\rangle - \langle\varphi, (\nabla_{e_i}e_i)\cdot\psi\rangle - \langle\varphi, e_i\cdot\nabla_{e_i}\psi\rangle\right)$$

ここで $V \in \mathfrak{X}(M)$ を

$$g(V, W) = -\langle\varphi, W\cdot\psi\rangle \quad \text{とする}$$

$$\mathrm{div}(V) = -\sum g(\nabla_{e_i} V, e_i)$$

$$= -\sum e_i \langle \varphi, e_i \psi \rangle + \langle \varphi, (\nabla_{e_i} e_i) \cdot \psi \rangle$$

$\therefore$ $\langle D\varphi, \psi \rangle = \mathrm{div}(V) + \langle \varphi, D\psi \rangle$

$\therefore$ div thm より, $\int_M \mathrm{vol}_g$ にて

$$(D\varphi, \psi) = (\varphi, D\psi)$$

<u>Cor</u> (M, g) cpt spin

$\Rightarrow$ $H(D) = H(D^2)$

$\because)$ $\subset$ は明らか. $D^2\varphi = 0$ なら

$$0 = (D^2\varphi, \varphi) = (D\varphi, D\varphi) = \|D\varphi\|^2$$

$\therefore$ $D\varphi = 0$ //

このように

$D : \Gamma(M, S) \to \Gamma(M, S)$ Dirac operator

- 1-st order elliptic
- formally self ad

$\rightsquigarrow$ cpt (M, g) 上で D のスペクトル分解ができる

Ex $M = \mathbb{R}/\mathbb{Z} = S^1$ (flat) $S = S^1 \times \mathbb{C}$, $D = -\sqrt{-1}\dfrac{\partial}{\partial\theta}$

$$L^2(M.\ S) = \overline{\bigoplus_{m\in\mathbb{Z}} E_m} \qquad E_m = \langle e^{im\theta} \rangle_{\mathbb{C}}$$

eigenvalue $= m$

$$f \in L^2(M.\ S),\quad f = \sum a_m e^{im\theta}$$

$$a_m = \int_{S^1} f(\theta)\, e^{-im\theta}\, d\theta \ //$$

この Ex の一般化が cpt spin mfd 上で可能

Thm $(M.g)$ cpt spin mfd

D のスペクトル分解 on $L^2(M.\ S) = \overline{\Gamma(M.\ S)}$

- $E_\lambda := \ker(D - \lambda\, id) \subset L^2(M.\ S)$

$$\begin{cases} \dim_{\mathbb{C}} E_\lambda = m(\lambda) < \infty \\ E_\lambda \text{ は smooth section からなる} \end{cases}$$

- $Spec(D) = \{\lambda \mid E_\lambda \neq \{0\}\} \subset \mathbb{R}$

  $Spec(D)$ discrete set, 上.下に非有界 $(\lambda \to \pm\infty)$

- $L^2(M.\ S) = \overline{\bigoplus_{\lambda\in Spec(D)} E_\lambda}$

  complete o.n. system $\{\phi_i\}$ がとれて.

  $\varphi \in L^2(M.S)$ は, $\varphi = \sum(\varphi.\ \phi_i)\phi_i$ //

$\dim M = 2m$

$$S = S^+ \oplus S^- \qquad X\cdot(S^\pm) \subset S^\mp$$
$$\nabla(S^\pm) \subset S^\pm$$

$$\therefore \quad \varphi \in \Gamma(M, S^\pm) \Rightarrow D\varphi \in \Gamma(M, S^\mp)$$

そこで $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$ on $\Gamma(M, S^+) \oplus \Gamma(M, S^-)$

$$D^2 = \begin{pmatrix} D^-D^+ & 0 \\ 0 & D^+D^- \end{pmatrix} \;//$$

<u>Note</u>

η-inv

$n \not\equiv 3 \mod 4 \Rightarrow \mathrm{Spec}(D)$ は原点対称

$\because)$ $n=\text{even}$

$$D\begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} = \lambda \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix}$$
$$\Leftrightarrow D\begin{pmatrix} \varphi_+ \\ -\varphi_- \end{pmatrix} = -\lambda \begin{pmatrix} \varphi_+ \\ -\varphi_- \end{pmatrix}$$

$J(\lambda\varphi) = \bar{\lambda} J\varphi$

$n \equiv 1 \mod 4$ $\quad$ $Cl_n \subset \mathbb{C}l_n$ の実現より. W_n に

$\mathrm{Spin}(n)$-eq & Cl_n の積と反可換な

real str or quaternion str J が入る

$\rightsquigarrow \ker D$ へ遺伝 $\quad D\varphi = \lambda\varphi \Rightarrow DJ\varphi = -\lambda J\varphi \;//$

§ Atiyah-Singer index (outline)

Heat operator

(M, g): cpt

P : formal selfadj elliptic op on $\Gamma(M, E)$

$\rightsquigarrow \{\phi_i\}$ c.o.n.s. s.t. $P\phi_i = \lambda_i \phi_i$

$0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots$ とする

$$\varphi = \sum (\varphi, \phi_i)\phi_i \in L^2(M, E), \quad t > 0$$

$$e^{-tP}(\varphi) := \sum_{i=1}^{\infty} e^{-t\lambda_i}(\varphi, \phi_i)\phi_i \text{ とする}$$

$t>0$ より, $i \to \infty$ で $e^{-t\lambda_i} \to 0$ (rapidly)

$\therefore e^{-tP}(\varphi) \in \Gamma(M, S)$ (C^∞-級!!)

$\rightsquigarrow$ 積分核

$$k_t(x, y) := \sum_i e^{-t\lambda_i} \phi_i(x) \otimes \overline{\phi_i(y)}$$

x, y について C^∞-class

$$e^{-tP}(\varphi) = \int_M k_t(x, y)\phi(y)\,\mathrm{vol}_g(y)$$

op. trace をとれて

$$\mathrm{Tr}(e^{-tP}) = \sum e^{-t\lambda_i}$$

$$= \int_M \mathrm{Tr}(\underbrace{k_t(x, x)}_{\in \mathrm{End}(E_x)})\,\mathrm{vol}_g$$

(M, g) $n = \text{even}$ spin mfd cpt

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \quad D^2 = \begin{pmatrix} D^-D^+ & 0 \\ 0 & D^+D^- \end{pmatrix}$$

まず $D^* = D$ より $(D^\pm)^* = D^\mp$

$H(D^2) = H(D)$ より. $\ker D^\pm = \ker D^\mp D^\pm$

さて. $D^-D^+ = (D^+)^* D^+$ nonnegative op

heat operator of D^-D^+, D^+D^- を考える

<u>Lem</u> $\mu \neq 0$,

$$D^+ : E_\mu(D^-D^+) \longrightarrow E_\mu(D^+D^-) \text{ isom}$$

$\because)$

$\phi_+ \in E_\mu(D^-D^+) \subset \Gamma(M, S^+)$.

$$D^+D^-(D^+\phi_+) = D^+(D^-D^+\phi_+) = \mu D^+\phi_+$$

$\therefore$ $D^+\phi_+ \in E_\mu(D^+D^-)$ となる

D^+の逆写像として $\frac{1}{\mu} D^- : E_\mu(D^+D^-) \to E_\mu(D^-D^+)$

をとればよい $_{//}$

ここで

$$\mathrm{tr}(e^{-tD^-D^+}) - \mathrm{tr}(e^{-tD^+D^-})$$

$$= \sum e^{-t\mu_+} - \sum e^{-t\mu_-}$$

0以外の固有値、重複度は一致

$$= \sum_{\mu_+ = 0} e^{-t\mu_+} - \sum_{\mu_- = 0} e^{-t\mu_-}$$

$$= \dim_{\mathbb{C}} \ker D^- D^+ - \dim_{\mathbb{C}} \ker D^+ D^-$$

$$= \dim_{\mathbb{C}} \ker D^+ - \dim_{\mathbb{C}} \ker D^-$$

(t に depend してない)

Def

$$\mathrm{ind}(D) := \dim_{\mathbb{C}} \ker D^+ - \dim_{\mathbb{C}} \ker D^-$$

を D の analytic index という。

$e^{-tD^-D^+}$, $e^{-tD^+D^-}$ の heat op をつかって

$$\mathrm{ind}(D) = \int_M \mathrm{tr}\left(k_t(x,x)^+ - k_t(x,x)^- \right) \mathrm{vol}_g$$

t によらない。

$t \to +0$ で右辺を漸近展開する

$$k_t(x,y)^+ - k_t(x,y)^- \sim \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{d(x,y)^2}{4t}\right)\left(\Theta_0(x,y) + t\,\Theta_1(x,y) + \cdots\right)$$

$$\therefore\ \mathrm{ind}(D) = \int_M \mathrm{tr}_x\left(\Theta_{\frac{n}{2}}(x,x)\right) \mathrm{vol}_g$$

積分項は

$\hat{A}(TM)$ という char class の n-form の part

に一致する!!

$\hat{A}(TM)$ は $\hat{A}$-class とよばれ

ポントリャーギン class の poly (see Milnor)

$$\hat{A}(TM) = 1 - \frac{1}{24} p_1(M) \quad \in H^4$$
$$+ \frac{1}{2^7 \cdot 3^2 \cdot 5} \left(-4 p_2(M) + 7 p_1(M)^2\right) \quad \in H^8$$
$$+ \cdots$$
$$\in H^*(M, \mathbb{R})$$

例 M cpt mfd 4-dim

$$\int_M \hat{A}(TM) = -\frac{1}{24} \int_M p_1(M)$$

Thm (index thm)

(M, g) cpt spin mfd

$$\Rightarrow \quad \mathrm{ind}(D) = \int_M \hat{A}(TM) =: \underbrace{\hat{A}(M)}_{\hat{A}\text{-genus}}$$

Note ind(D) は, g によらない diff top invariant //

Cor M spin str がある $\Rightarrow$ $\hat{A}(M) \in \mathbb{Z}$ //

Prop M : $4k$-dim cpt mfd

s.t. $\hat{A}(M) \neq 0$, $w_1(M) = w_2(M) = 0$

$\Rightarrow$ M には Scal>0 となる g は入らない.

∵) もし, そのような g が入るなら,

(M, g) spin mfd with Scal>0

vanishing thm より $H(D) = \{0\}$

$0 = \dim\ker D^+ - \dim\ker D^- = \hat{A}(M)$ 矛盾 //

Note $\dim\ker D^+ - \dim\ker D^-$ は diff top inv

一方 $\dim H(D) = \dim\ker D^+ + \dim\ker D^-$

は g, spin str に depend //

有名な応用 (ロホリンの定理)

Thm M cpt 4-dim mfd $w_1(M) = w_2(M) = 0$

$\Rightarrow$ $\sigma(M)$ は 16 の倍数 //

proof (M, g) cpt 4-dim spin mfd

なら $\dim\ker D^{\pm} = \text{even}$ を示す.

$Cl_4 \cong \mathbb{H}(2) \curvearrowright \mathbb{H}^2 \cong \mathbb{C}^4$

$Cl_4 \times \mathbb{C}^4 \ni (\phi \otimes z, v) \longmapsto \phi \cdot v \cdot z \in \mathbb{C}^4$

$\therefore\ W_4 \cong \mathbb{H}^2 \cong \mathbb{C}^4,\ W_4^{\pm} \cong \mathbb{H} \cong \mathbb{C}^2$

$\mathbb{H} \curvearrowleft \quad \mathbb{H} \times \mathbb{H} \ni (p, \nu) \mapsto \nu \bar{p} \in \mathbb{H}$

により，四元数 Str が入り，明らかに

これは Spin(4)-eq である

$\rightsquigarrow$ q-Str on $S^{\pm}$, compatible with ∇

$\rightsquigarrow$ $\ker(D^{\pm})$ にも q-Str が入る　　$\nearrow$ M^{8k+4} でも O.K.

$\therefore\ \dim_{\mathbb{C}} \ker(D^{\pm}) = \text{even}$ //

A-S-index thm より，

alg top の fact

$$\hat{A}(M) = -\frac{1}{24}\int p_1(M) = -\frac{1}{8}\sigma(M)$$

が even となる

ここで，$Q : H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \to \mathbb{R}$

$$(\varphi, \psi) \mapsto \int_M \varphi \wedge \psi$$

は $H^2(M, \mathbb{R})$ 上対称2次形式

$\sigma(M) = \text{sign}(Q)$ //

<u>Fact</u> $\exists$ 4-dim topo mfd M_0 s.t.

$\sigma(M_0) = 8,\ w_1(M_0) = w_2(M_0) = 0$

$\rightsquigarrow$ この M_0 に diff str は入らない //

§ Eigenvalue estimate

$(M.g)$ cpt spin mfd

$$D^2 = \underbrace{\nabla^*\nabla}_{\geqq 0} + \frac{1}{4}\mathrm{Scal} \quad , \quad \lambda \in \mathrm{Spec}(D)$$

$$\therefore \lambda^2 \geqq \frac{1}{4} \min_{x\in M} \mathrm{Scal}(x)$$

もっとよい評価がある。

S, TM associated bdle of $\mathrm{Spin}(M)$

$S \otimes TM$ 〃

$$S \otimes TM \cong S \oplus S_{3/2} \quad (W_n \otimes_{\mathbb{C}} \mathbb{R}^n \cong W_n \oplus T_n)$$

<u>Note</u> h.w of Spin(n)

$$(\tfrac{1}{2}, \cdots, \tfrac{1}{2}) \otimes (1, 0 \cdots 0) = (\tfrac{1}{2}, \cdots \tfrac{1}{2}) \oplus (\tfrac{3}{2}, \tfrac{1}{2}, \cdots \tfrac{1}{2})$$

<u>ex</u> $\mathrm{Spin}(3) = SU(2)$

$SU(2) \curvearrowright \mathbb{C}^2 \cong V_1$ spin $1/2$-rep

$SU(2) \curvearrowright S^k(\mathbb{C}^2) \cong V_k$ spin $k/2$ rep

($k+1$-dim)

このとき $V_1 \otimes V_2 \cong V_1 \oplus V_3$

Clebsch-Gordan

$W_n \otimes \mathbb{R}^n \cong W_n \oplus T_n$ を与えよう.

$\Pi : W_n \otimes \mathbb{R}^n \ni \phi \otimes v \longmapsto v \cdot \phi \in W_n$ は $Spin(n)$-eq

$\because)$ $\Pi(g(\phi \otimes v)) = \Pi(g\phi \otimes Ad(g)v)$

$= (gvg^{-1}) \cdot (g\phi) = g \cdot v \cdot \phi = g \cdot \Pi(\phi \otimes v)$ //

$\therefore$ $T_n := \ker \Pi$ は $Spin(n)$-inv subsp

また $\iota : W_n \ni \phi \longmapsto -\frac{1}{n} \sum e_i \cdot \phi \otimes e_i \in W_n \otimes_{\mathbb{C}} \mathbb{R}^n$

も $Spin(n)$-eq, inj map

s.t. $\Pi \circ \iota = id$

$\therefore$ $W_n \otimes \mathbb{R}^n \cong W_n \oplus T_n$ (irreducible decomp)

具体的には

$$\varphi \otimes v = -\frac{1}{n} \sum e_i \cdot v \cdot \varphi \otimes e_i$$

$$+ \left(\varphi \otimes v + \frac{1}{n} \sum e_i \cdot v \cdot \varphi \otimes e_i\right)$$

また $\varphi = \sum \varphi_i \otimes e_i \in W_n \otimes \mathbb{R}^n$

$\varphi \in T_n = \ker \Pi \iff \sum e_i \cdot \varphi_i = 0$ //

<u>Note</u> $n =$ even

$$W_n^{\pm} \otimes \mathbb{R}^n \cong T_n^{\pm} \oplus W_n^{\mp}$$

すべて Spin(n)-eq より

$\rightsquigarrow\ S\otimes T^*M \cong S\otimes TM \cong S\oplus S_{3/2}$

$S_{3/2} := \mathrm{Spin}(M)\times_{\rho} \ker\pi$

twistor bundle という

$\nabla\varphi = \sum \nabla_{e_i}\varphi\otimes e_i$ （∇ も 分解を保つ）

$\nabla : \Gamma(S) \to \Gamma(S\otimes T^*M)$, $T^*M \cong TM$, $S\otimes T^*M \cong S\oplus S_{3/2}$

prop

$$D = \sum e_i\cdot\nabla_{e_i} = \pi\circ\nabla$$

$$(\pi : S\otimes TM \to S \ \text{proj})$$

では. $\pi_{3/2} : S\otimes TM \to S_{3/2}$ は??

Def Penrose (twistor) operator

$$P := \pi_{3/2}\circ\nabla : \Gamma(M, S) \to \Gamma(M, S_{3/2})$$

explicitly,

$$P(\varphi) = \pi_{3/2}\left(\sum \nabla_{e_i}\varphi\otimes e_i\right)$$

$$= \sum\left(\nabla_{e_i}\varphi + \frac{1}{n}e_i\cdot D\varphi\right)\otimes e_i$$

$\varphi\in\ker P = \{\varphi\in\Gamma(M, S) \mid P(\varphi) = 0\}$

を twistor spinor という。

Note $n = \text{even}$ $P^{\pm} : \Gamma(M, S^{\pm}) \to \Gamma(M, S^{\pm}_{3/2})$

P の (formal) adjoint は?

i.e. $(P\varphi, \psi) = (\varphi, P^*\psi)$

($\varphi \in \Gamma(M, S)$, $\psi \in \Gamma(M, S_{3/2})$)

どちらか compact support

となる $P^* : \Gamma(M, S_{3/2}) \to \Gamma(M, S)$

<u>prop</u> P の formal adj op P^* は

$$P^* = -\sum \underline{c_{TM}(e_i)}\, \nabla_{e_i} \quad \text{on } \Gamma(M, S_{3/2})$$

TM part の ~~tensor~~ interior product

explicitly

for $\varphi = \sum \varphi_i \otimes e_i$ s.t. $\sum e_i \cdot \varphi_i = 0$,

$$P^*(\varphi) = -\sum c_{TM}(e_i)\nabla_{e_i}(\varphi_j \otimes e_j)$$

$$= -\sum c_{TM}(e_i)\left((\nabla_{e_i}\varphi_j) \otimes \underline{e_j} + \varphi_j \otimes \underline{\nabla_{e_i} e_j}\right)$$

$$= -\sum \nabla_{e_i}\varphi_i - \sum g(e_i, \nabla_{e_i} e_j)\varphi_j$$

<u>proof</u> $\varphi = \sum \varphi_i \otimes e_i \in \Gamma(S_{3/2})$, $\psi \in \Gamma(S)$

$$\langle P\psi, \varphi\rangle = \langle \sum(\nabla_{e_j}\psi + \tfrac{1}{n} e_j D\psi) \otimes \underline{e_j}, \sum \varphi_i \otimes \underline{e_i}\rangle$$

δ_{ij}

$$= \sum \langle \nabla_{e_i}\psi, \varphi_i\rangle + \frac{1}{n}\sum\langle e_i D\psi, \varphi_i\rangle$$

$\hat{e_i}$

$\sum e_i \varphi_i = 0$

$$= \sum e_i\langle\psi, \varphi_i\rangle - \sum\langle\psi, \nabla_{e_i}\varphi_i\rangle$$

$V = \sum \langle \psi, \varphi_i \rangle e_i \in \mathfrak{X}(M)$ とすると

$= \mathrm{div}(V) + \langle \psi, P^*(\varphi) \rangle$ となる

あとは $\int_M \mathrm{vol}_g$ をとればよい。

$\nabla \sim D + P$ 　実際

<u>Prop</u>

$$\nabla^* \nabla = \frac{1}{n} D^2 + P^* P$$

<u>Cor</u> 　(M, g) cpt spin mfd

$\{$parallel spinors$\} = \ker \nabla$

$= \{$twistor spinors$\} \cap \{$harmonic spinors$\}$

$\{$twistor spinors$\} = \ker P$, $\{$harmonic spinors$\} = \ker D$

<u>proof</u>

$$P^* P(\varphi) = -\sum \nabla_{e_i} \left(\nabla_{e_i} \varphi + \frac{1}{n} e_i D\varphi \right)$$

$$- g(e_i, \nabla_{e_i} e_j) \left(\nabla_{e_j} \varphi + \frac{1}{n} e_j D\varphi \right)$$

$$= -\sum \left(\nabla_{e_i} \nabla_{e_i} \varphi - \nabla_{\nabla_{e_i} e_i} \varphi \right)$$

$$- \frac{1}{n} \sum \left((\nabla_{e_i} e_i) D\varphi + e_i \nabla_{e_i} (D\varphi) \right)$$

$$+ \frac{1}{n} \sum (\nabla_{e_i} e_i) D\varphi$$

$$= \nabla^* \nabla \varphi - \frac{1}{n} D^2 \varphi$$ 〃

Thm (Weitzenböck type formula)

$$\frac{1}{2} P^*P + \frac{-(n-1)}{2} \frac{1}{n} D^2 = -\frac{Scal}{8}$$

(意) geometric second order diff op の "よい linear combi" は 曲率 になる

Ex

$$-C^*C + \frac{1}{2}\delta d + \frac{n-1}{n} d\delta = Ric$$

(☆)

↑ Conformal killing op

proof of thm

$$D^2 = \nabla^*\nabla + \frac{1}{4}Scal$$
$$= P^*P + \frac{1}{n}D^2 + \frac{1}{4}Scal \quad //$$

Note "$\frac{1}{2}$", "$-\frac{(n-1)}{2}$" は conformal weight

$$g \rightsquigarrow g' = e^{2\sigma}g$$

$$\Rightarrow \quad D' = e^{(-\frac{n-1}{2}-1)\sigma} \circ D \circ e^{\frac{n-1}{2}\sigma}$$

$$P' = e^{(\frac{1}{2}-1)\sigma} \circ P \circ e^{-\frac{1}{2}\sigma}$$

Thm (Friedrich's estimate)

(M^n, g) cpt spin mfd, $\lambda \in Spec(D)$

$$\lambda^2 \geqq \frac{n}{4(n-1)} \min_{x\in M} Scal(x)$$

proof　$D\varphi = \lambda\varphi$ とする

$$\lambda^2 (\varphi, \varphi) = (D^2\varphi, \varphi)$$

$$= \frac{n}{n-1} \underset{\|P\varphi\|^2 \geq 0}{(P^* P \varphi, \varphi)} + \frac{n}{4(n-1)} \int_M \langle \mathrm{Scal}\, \varphi, \varphi \rangle \mathrm{vol}_g$$

$$\geq \frac{n}{4(n-1)} \min_{x\in M} \mathrm{Scal}(x) \|\varphi\|^2$$

//

Note　W-B formula は vanishing thm や (下からの) eigenvalue estimate を与える。

Question

$$\lambda^2 = \frac{n}{4(n-1)} \overbrace{\min \mathrm{Scal}}^{= C_0}$$ をみたす spin mfd は存在するか?

存在するなら estimate は sharp といえる。

"limiting mfd" という。

"=" $\Leftrightarrow$ $\exists \varphi \in \Gamma(M, S)$ s.t.

$$P(\varphi) = 0 \ \& \ D\varphi = \pm\sqrt{\frac{n}{4(n-1)} C_0}\, \varphi$$

Exc　$P(\varphi) = 0$

$$\Leftrightarrow \nabla_X \varphi + \frac{1}{n} X \cdot D\varphi = 0 \quad (\forall X \in \mathfrak{X}(M))$$

$$\Leftrightarrow X \cdot \nabla_Y \varphi + Y \cdot \nabla_X \varphi = \frac{2}{n} g(X, Y) D\varphi \quad \forall X, Y$$

//

$\lambda = \gamma$ "$=$" $\Leftrightarrow$ $^\exists \varphi \in \Gamma(M, S)$ s.t.

$$\nabla_X \varphi = \pm \sqrt{\frac{n}{4(n-1)} c_0}\ X \cdot \varphi \quad \forall X \in \mathfrak{X}(M)$$

Def (M, g) spin mfd

$\varphi \in \Gamma(M, S)$ が killing number $\mu \in \mathbb{C}$ の killing spinor とは

$$\nabla_X \varphi = \mu X \cdot \varphi \quad \forall X \in \mathfrak{X}(M)$$

Note ・ $\mu = 0 \Rightarrow \varphi$ parallel spinor

・ $n = \text{even}$

$\varphi = \varphi_+ + \varphi_-$ killing spinor with μ

$\Rightarrow \tilde{\varphi} = \varphi_+ - \varphi_-$ 〃 $-\mu$

prop. φ killing spinor $\Rightarrow$ φ twistor spinor

proof $D\varphi = \sum e_i \nabla_{e_i} \varphi = \sum \mu e_i e_i \varphi$

$= -n\mu\varphi$

$\therefore \nabla_X \varphi + \frac{1}{n} X \cdot D\varphi = 0$ //

prop φ killing spinor, $\mu \in \mathbb{R}$

$\Rightarrow V^\varphi = \sqrt{-1} \sum \langle e_i \varphi, \varphi \rangle e_i$ is killing vector field //

killing vector field X

$\iff$ X の flow が isometry

$\iff$ $L_X g = 0$

$\iff$ $X(g(Y,Z)) = g([X,Y],Z) + g(Y,[X,Z])$

$\iff$ $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$

<u>proof of prop</u>

$$\overline{V^\varphi} = V^\varphi \quad \therefore \quad V^\varphi \in \mathfrak{X}(M)$$

さて、 $\nabla_X V^\varphi = \sqrt{-1} \sum \big(\langle (\nabla_X e_i)\varphi, \varphi \rangle e_i$

$+ \langle e_i \underbrace{\nabla_X \varphi}_{= \mu X \cdot \varphi}, \varphi \rangle e_i + \langle e_i \varphi, \underbrace{\nabla_X \varphi}_{= \mu X \cdot \varphi} \rangle e_i$

$\mu \in \mathbb{R}$ $\quad + \langle e_i \varphi, \varphi \rangle \nabla_X e_i \big)$

$= \sqrt{-1} \mu \sum \langle (e_i \cdot X - X \cdot e_i \cdot)\varphi, \varphi \rangle e_i$

$+ \sqrt{-1} \sum \big(\langle (\nabla_X e_i) \cdot \varphi, \varphi \rangle e_i + \langle e_i \cdot \varphi, \varphi \rangle \nabla_X e_i \big)$

$g(\nabla_X V^\varphi, Y)$

$= \sqrt{-1} \mu \langle (Y \cdot X - X \cdot Y \cdot)\varphi, \varphi \rangle + 0$

$\therefore \quad g(\nabla_X V^\varphi, Y) + g(X, \nabla_Y V^\varphi) = 0$

Thm (M^n, g) conn spin mfd with killing spinor φ with killing number μ.

$\Rightarrow$ (1) (M, g) is Einstein manifold

(2) $\mu^2 = \frac{1}{4} \frac{1}{n(n-1)} \mathrm{Scal}$ $\quad$ $\nabla_X \varphi = \mu X \cdot \varphi$ ($\forall X$)

よ, 2 $\mu \in \mathbb{R}$ or $\sqrt{-1}\,\mathbb{R}$

他の killing number があるなら $-\mu$.

(3) (3-1) $\mu \in \mathbb{R}$ ($\mu \neq 0$) のとき

φ は real killing spinor という

- (M, g) positive Einstein, $\quad$ complete $\Rightarrow$ cpt
- $D\varphi = \pm \frac{1}{2} \sqrt{\frac{n}{n-1} \mathrm{Scal}} \cdot \varphi$
- $\langle \varphi, \varphi \rangle = \mathrm{const}$.
- V^φ is killing vector

(3-2) $\mu \in \sqrt{-1}\,\mathbb{R}$ ($\mu \neq 0$) のとき

φ は imaginary killing spinor という

- (M, g) : non cpt negative Einstein
- $\langle \varphi, \varphi \rangle$ は non const fn で zero 点なし.
- V^φ は conformal vector field

(3-3) $\mu = 0$ のとき

φ は parallel spinor

・(M, g) は Ricci flat ($Ric = 0$)

・V^{φ} は killing vector field 〃

Lem

φ killing spinor

$\nabla'_X := \nabla_X - \mu X \cdot$

φ は ∇'- parallel

とくに φ に zero 点 は ない 〃

proof of thm

$$\nabla_Y \varphi = \mu Y \cdot \varphi \quad (\forall Y \in \mathfrak{X}(M))$$

$$\nabla_X \nabla_Y \varphi = \mu (\nabla_X Y) \cdot \varphi + \mu^2 Y \cdot X \cdot \varphi$$

$$\therefore \ R_\Delta(X, Y)\varphi = \mu^2 (Y \cdot X \cdot - X \cdot Y \cdot)\varphi$$

ここで $\nabla_X Y - \nabla_Y X - [X, Y] = 0$ を用いた.

$$-\frac{1}{2} Ric(X) \cdot \varphi = \sum e_i \cdot R_\Delta(X, e_i)\varphi$$

$$= \sum \mu^2 e_i \cdot (e_i \cdot X \cdot - X \cdot e_i \cdot)\varphi$$

$$= 2(1-n)\mu^2 X \cdot \varphi$$

$$\therefore \ (Ric(X) - 4(n-1)\mu^2 X) \cdot \varphi = 0$$

$$\therefore \quad 0 = \sum e_i \left(Ric(e_i) - 4(n-1)\mu^2 e_i \right) \cdot \varphi$$

$$= \left(-Scal + 4n(n-1)\mu^2 \right) \varphi$$

φ に zero 点 なし $\therefore$

$$Scal = 4n(n-1)\mu^2$$

$\therefore$ $Scal = const$, $\mu \in \mathbb{R}$ or $\sqrt{-1}\,\mathbb{R}$

$(Scal \geqq 0)$ $(Scal < 0)$

さらで $\left(Ric(X) - 4(n-1)\mu^2 X \right) \cdot \varphi = 0$

$$\left(\begin{array}{l} Y \cdot \varphi = 0 \Rightarrow \|Y\|^2 \varphi = 0 \\ \varphi \text{ zero 点がないので } \quad Y = 0 \end{array} \right)$$

$$Ric(X) = 4(n-1)\mu^2 X = \frac{Scal}{n} X$$

$\therefore$ $Ric = cg$ より $(M.g)$ Einstein

$\mu = 0 \Rightarrow Ric = 0$ (3.3) o.k.

$\mu \in \mathbb{R}(\neq 0) \Rightarrow Scal > 0$ $\quad Ric = \frac{Scal}{n} g$

$(M\ g)$ complete なら マイヤース定理より

$(M.g)$ cpt

また $D\varphi = \sum e_i \nabla_{e_i} \varphi = -n\mu\varphi$

$$= \pm \frac{1}{2} \sqrt{\frac{Scal}{n(n-1)}} \cdot \varphi$$

とくに estimate で "=" 成立

また $X\langle\varphi,\varphi\rangle = \langle\nabla_X\varphi,\varphi\rangle + \langle\varphi,\nabla_X\varphi\rangle$

$$= (\mu-\bar{\mu})\langle X\cdot\varphi,\varphi\rangle$$

$\mu\in\mathbb{R}$ より $\langle\varphi,\varphi\rangle = const.$

∴ (3-1) o.k.

(3-2) をみてみる. $\mu=\sqrt{-1}\,b$ $(b\in\mathbb{R}\neq 0)$

$$Scal = 4n(n-1)\mu^2<0$$

M cpt とすると $D^2\geqq 0$ なので $D^2\varphi=-n^2b^2\varphi$

∴ M noncpt.

さて $X\langle\varphi,\varphi\rangle = (\mu-\bar{\mu})\langle X\cdot\varphi,\varphi\rangle$

$$= 2\sqrt{-1}\,b\langle X\cdot\varphi,\varphi\rangle$$

$\left(V^\varphi = \sqrt{-1}\sum\langle e_i\cdot\varphi,\varphi\rangle e_i \text{ より}\right)$

$$= 2b\, g(V^\varphi, X)$$

ここで $L_{V^\varphi}g = 4b|\varphi|^2 g$ を示す

∵) $\nabla_X V^\varphi = \cdots = -b\sum\langle(e_i\cdot X\cdot + X\cdot e_i\cdot)\varphi,\varphi\rangle e_i$

$$= 2b|\varphi|^2\sum g(X,e_i)e_i$$

$$= 2b|\varphi|^2 X$$

∴ $(L_{V^\varphi}g)(Y,Z) = g(\nabla_Y V^\varphi, Z) + g(Y,\nabla_Z V^\varphi)$

$$= 4b|\varphi|^2 g(Y,Z) \quad //$$

そこで $\langle \varphi, \varphi \rangle = \mathrm{const}\ (\neq 0)$ とすると

$0 = X\langle \varphi, \varphi \rangle = 2b\, g(V^{\varphi}, X)$ より

$V^{\varphi} = 0$ $\quad \therefore \quad 4b|\varphi|^2 = 0$ 矛盾

$\therefore$ $\langle \varphi, \varphi \rangle$ は noncont fn で

Lem より zero 点もない

また $L_{V^{\varphi}} g = {}^{\exists}f \cdot g \quad f \in C^{\infty}(M)$

$\therefore$ V^{φ} は conformal vector field //

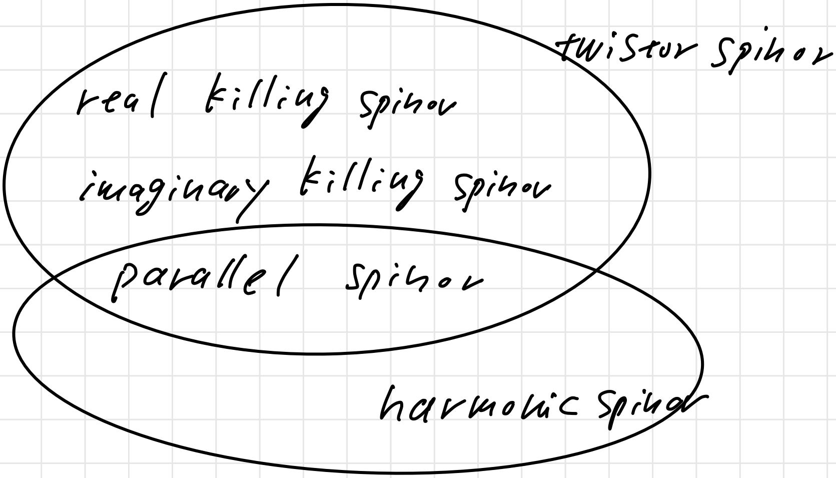
<u>Note</u> X: conformal vector field

$\Leftrightarrow$ X の flow が conformal transf

i.e. $\phi_t^* g = e^{2t\lambda} g \quad {}^{\exists}\lambda \in C^{\infty}(M)$

$\Leftrightarrow$ $L_X g = 2\lambda g$

<u>Exc</u> $L_X g = 2\lambda g \Rightarrow \lambda = \mathrm{div}(X)/n$



§ Killing spinors

$$\nabla_X \varphi = \mu X \cdot \varphi \qquad \forall X \in \mathfrak{X}(M)$$

$\Rightarrow (M, g)$ Einstein, $Scal = 4n(n-1)\mu^2$

metricを $g \mapsto ag$ $(a > 0)$ とかれば

$\mu = \pm\frac{1}{2}, \ 0, \ \pm\frac{\sqrt{-1}}{2}$ に normalize.

そこで

$$K_\pm := \{ \varphi \in \Gamma(M, S) \mid \nabla_X \varphi = \pm\frac{1}{2} X \cdot \varphi \}$$

(or $\pm\frac{\sqrt{-1}}{2}$)

とおく.

$\dim M = \text{even}$ $\qquad \varphi = \varphi_+ + \varphi_- \longrightarrow \tilde{\varphi} = \varphi_+ - \varphi_-$

により $\qquad K_+ \cong K_-$

また $\quad \varphi$ Killing $\Leftrightarrow$ $\nabla'_X = \nabla_X - \mu X \cdot$ parallel

prop

$$\dim_{\mathbb{C}} K_\pm \leq 2^{[\frac{n}{2}]} = \dim W_n \quad //$$

dim M = 2 $\qquad R$ は scalar曲率 = ガウス曲率

$\therefore (M^2, g)$ with Killing $\Rightarrow$ 定曲率

<u>dim $M=3$</u> $\quad Ric = cg$ より,

(M^3, g) 定曲率空間

<u>Note</u> (M^3, g) with twistor spinor φ

$\Rightarrow$ (M^3, g) conformally flat.

<u>dim $M=4$</u> (see Fridrichの本)

(M^4, g) φ killing $(\mu \neq 0)$

$\Rightarrow (M^4, g)$ 定曲率

$\left(\mu = \pm\frac{1}{2} \Rightarrow (M^4, g) \cong (S^4, g_0) \right)$

φ parallel, M^4 cpt spin

$\Rightarrow$ (M^4, g) K3-surface or flat torus

($Hol(M) = SU(2)$ あとで)

<u>Q</u> dim $M \geqq 5$ killing spinor をもつ mfd は??

その前にいくつかThm

<u>Thm</u> (M^n, g) spin with killing spinor $(\mu \neq 0)$

$\Rightarrow$ (M^n, g) irreducible as Riem mfd

<u>Note</u>

(M_i, g_i) Riem mfd $\quad i = 1, 2$

$M = M_1 \times M_2$

$g = g_1 \times g_2$ $\qquad$ $T_x M = T_{x_1} M_1 \oplus T_{x_2} M_2$

$g(v_1 + v_2, w_1 + w_2)$

$= g_1(v_1, w_1) + g_2(v_2, w_2)$

この (M, g) を $(M_1 g_1), (M_2 g_2)$ の Riem product という

<u>Def</u> $(M.g)$ locally Riem product のとき

locally reducible といい、 そうでないとき

$(M.g)$ irreducible という。

<u>Exc</u>

$R(v_1 + v_2, w_1 + w_2)(u_1 + u_2)$

$= R_1(v_1, w_1)u_1 + R_2(v_2, w_2)u_2$ を示せ。

<u>proof of thm</u>

φ killing とすると

$R_\Delta(X.Y)\varphi = \mu^2(Y\cdot X\cdot - X\cdot Y\cdot)\varphi$

M locally reducible とすると、

$X \in T_{x_1}U_1$, $Y \in T_{x_2}U_2$ $\qquad$ $U = U_1 \times U_2 \subset M$

$R_\Delta(X.Y) = 0$, $g(X.Y) = 0$

$\therefore$ $\mu \neq 0$ より $X\cdot Y\cdot \varphi = 0$

$0 = X \cdot X \cdot Y \cdot \varphi = -g(X, X)\, Y \cdot \varphi$

$\therefore\ Y \cdot \varphi = 0 \qquad g(Y, Y)\, \varphi = 0$ on U

φ に zero 点 はないのであった。矛盾。

<u>prop</u>

(M^n, g) spin with killing spinor $(\mu \neq 0)$

$\Rightarrow$ parallel p-form は存在しない ($p \neq 0, n$)

とくに (M, g) に kähler str, q. kähler str

G_2-str, Spin(7)-str は入らない

<u>out line</u>

ω : p-form

$$D(\omega \cdot \varphi) = (-1)^p \omega \cdot (D\varphi) + ((d+\delta)\omega) \cdot \varphi - 2 \sum (\iota(e_i)\omega) \cdot \nabla_{e_i} \varphi$$

φ killing $\nabla \omega = 0$ なら

$$D^2(\omega \cdot \varphi) = \mu^2 (n-2p)^2\, \omega \cdot \varphi$$

$\lambda^2 \geq n^2 \mu^2$ より $p \neq 0, n$ なら $\omega \cdot \varphi = 0$

$0 = \nabla_X(\omega \cdot \varphi) = \mu\, \omega\, X \cdot \varphi \qquad \therefore\ \omega \cdot X \cdot \varphi = 0$

$\Rightarrow (\iota(X) \cdot \omega) \cdot \varphi = 0$

くり返して $\omega(X_1, \cdots X_p)\cdot\varphi = 0$

φ zero点 なし $\therefore$ $\omega = 0$

kähler parallel 2-form ω

q. kähler 〃 4-form Φ

G_2-mfd parallel 3-form φ

Spin(7) mfd 〃 4 〃 Ψ //

<u>prop</u>

(M, g) locally symm ($\nabla R = 0$) spin.

with killing spinor φ

$\Rightarrow (M, g)$ 定曲率

<u>proof</u> 略

(M, g) with killing or parallel spinor

をみたい.

・sym space 除いてよい.

・(M, g) irr としてよい

・$\exists (\tilde{M}, \tilde{g}) \xrightarrow[\pi]{} (M, g)$ universal covering, local isometry

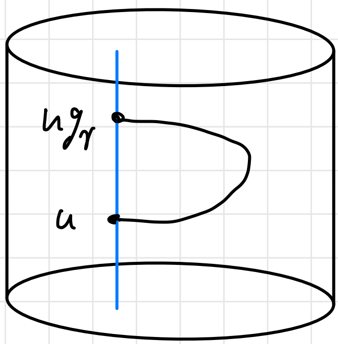
(M, g) simply conn としてよい.

§ Holonomy と Berger の list

(M, g) conn Riem mfd

$$\begin{array}{c} P = O(M) \curvearrowleft O(n) = G \\ \downarrow \\ M \end{array}$$

A_{LC} conn 1-form



γ : curve $x \overset{\gamma}{\sim} y$

$\Phi(\gamma) : P_x \longrightarrow P_y$

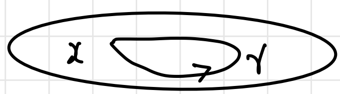
G-eq diffeo

$x \in M$ fix , $u \in P_x$ fix

$\Omega_x(M) = \{ \gamma : I \to M \mid \gamma(0) = \gamma(1) = x \}$

$\quad \cup\!\!\!\!\!\text{ } \ni$

$\quad \gamma$



$\Phi(\gamma)(u) = u\, {}^{\exists} g_\gamma \quad (g_\gamma \in G)$

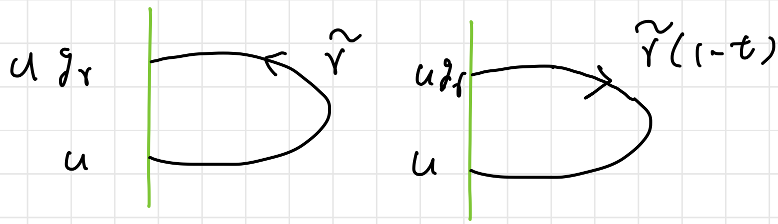
$\Psi_u : \Omega_x(M) \ni \gamma \longmapsto g_\gamma \in G$

<u>Lem</u> $\Psi_u(\Omega_x(M))$ は G の subgroup.

$\because)$ $\Phi(\gamma)(u) = u g_\gamma \quad \Phi(\gamma')(u) = u g_{\gamma'}$

$$\Phi(\gamma\gamma')(u) = \Phi(\gamma)(\Phi(\gamma')(u)) = \Phi(\gamma)(u g_{\gamma'})$$
$$= \Phi(\gamma)(u) g_{\gamma'} = (u g_\gamma) g_{\gamma'}$$
$$= u(g_\gamma g_{\gamma'}) = u g_{\gamma \circ \gamma'}$$
$$\therefore\ g_\gamma g_{\gamma'} \in \Psi_u(\Omega_x(M))$$

$\gamma_-(t) := \gamma(1-t)$ の horizontal lift は $\tilde{\gamma}(1-t)$



$$\Psi(\gamma_-)(u) = u g_{\gamma_-}$$
$$\Psi(\gamma_-)(u) g_\gamma = \Psi(\gamma_-)(u g_\gamma) = u$$
$$\therefore\ (u g_{\gamma_-}) g_\gamma = u \qquad \therefore\ g_{\gamma_-} = g_\gamma^{-1} \in \Psi_u(\Omega_x(M))$$

Def

$$\mathrm{Hol}(M,g) := \Psi_u(\Omega_x(M)) \subset O(n)$$

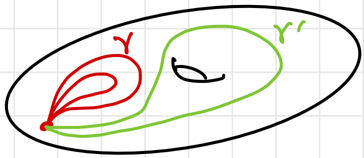
Riem holonomy group (ホロノミー群)

$$\mathrm{Hol}^0(M,g) := \Psi_u(\Omega_x^0(M))$$

restricted Riem holonomy group

という。 $\Omega_x^0(M) = \{\gamma \in \Omega_x^0(M) \mid \gamma \sim x \text{ const loop}\}$

$\gamma \in \Omega_x^0(M) \not\ni \gamma'$

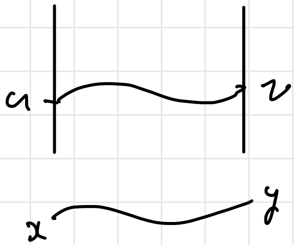


<u>Note</u> $u \rightsquigarrow ug$

$$\Psi_{ug}(\Omega_x(M)) = g^{-1}\Psi_u(\Omega_x(M))g$$

$x \rightsquigarrow y$ base pt change にて

$$\Psi_u(\Omega_x(M)) = \Psi_v(\Omega_y(M))$$



<u>Fact</u>

- $Hol(M)$ is Lie subgr of $O(n)$
- $Hol^0(M)$ is id conn comp of $Hol(M)$
- $\mathfrak{h}(M) := Lie(Hol(M))$

  Lie subalg of $\mathfrak{o}(n)$ //

<u>prop</u> $(F_{A_{LC}})_u$ は $\mathfrak{h}(M)$ に値をもつ

(R は $\mathfrak{h}(M)$-valued 2-form)

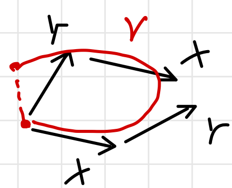
<u>outline of proof</u>

X, Y horizontal vector field on $P = O(M)$

ϕ_t ψ_t flow

$$\gamma(t) = \psi_{-s}\circ\phi_{-s}\circ\psi_s\circ\phi_s(u)$$

$s = \sqrt{t}$



$\gamma'(0) = [X, Y]_u \in \mathfrak{g}(M)$

F_A は X, Y horizontal vector field の $[X, Y]$ の vertical 成分を測る

$$F_A(X, Y) := [X, Y]_u \in \mathfrak{g}(M)$$

<u>Ex</u> (M, g) kähler mfd $(n = 2m)$

R は $\mathfrak{u}(m)$-valued 2-form

<u>Thm</u> (holonomy Reduction)

$$\begin{array}{c} O(M) \curvearrowleft O(n) \\ \downarrow \\ M \end{array}$$

$\Rightarrow$ $\exists$ $\begin{array}{c} Q \curvearrowleft Hol(M) \\ \downarrow \\ M \end{array}$ s.t.

$$\begin{tikzcd} Q \arrow[r, hook, "i"] \arrow[dr] & O(M) \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\ & M \end{tikzcd}$$

$Hol(M)$-eq map

$(i(ug) = i(u)g)$

& $i^* A_{LC}$ は

Q 上の conn 1-form

そして，Q，$i^* A_{LC}$ から P と A_{LC} を復元できる。

$P = O(M)$

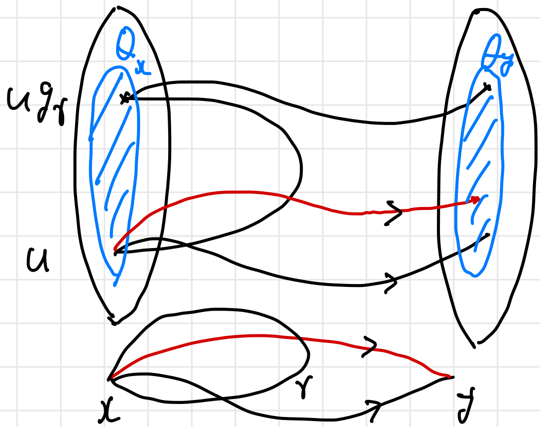
<u>outline of proof</u>

$u \in P$ fix

$Q = \{ v \mid {}^{\exists}\gamma \text{ curve in } P,\ \gamma(0)=u,\ \gamma(1)=v$

$\gamma'(t) \in H_{\gamma(t)} \} \subset P$

（つまり、u と horizontal curve で結べる点全体）



Q は M 上 pr $Hol(M)$-bundle.

また、$TP = H \oplus V$ とすると

$TQ = H \oplus (V \cap TQ)$ により、Q 上 conn を得るが、これは $i : Q \longrightarrow P$ を考えて i^*A に対応する。

$\iota : Hol(M) \to G$

また、Q から $P := Q \times_\iota O(n)$

P を再現でき、H が定まっているので conn on P を得る。

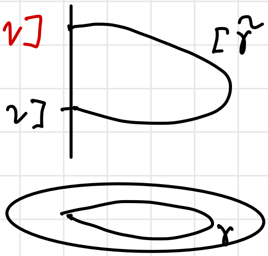
parallel section と holonomy

$$(M, g) \qquad \begin{array}{c} O(M) \circlearrowleft O(n) \\ \downarrow \\ M \end{array} \quad \rightsquigarrow \quad \begin{array}{c} Q \circlearrowleft Hol(M) \\ \downarrow \\ M \end{array}$$

$$\begin{array}{c} \mathbb{V} = O(M) \times_\rho V \\ \downarrow \\ M \end{array} \quad \text{with } \nabla$$

$[u, \rho(\theta_\gamma) v]$ $\qquad$ $[\tilde{\gamma} . v]$

$\varphi = [u . v]$



holonomy gr が def でき

$$Hol(\nabla) = \rho(Hol(M)) \subset GL(V)$$

prop $\qquad x_0 \in M$ fix

$s \in \Gamma(M, \mathbb{V})$, $\nabla s = 0$

$\Rightarrow$ $s(x_0)$ は $\rho(Hol(M))$-inv vector //

逆に $v \in V$ が $\rho(Hol(M))$-inv vector

$\Rightarrow$ parallel section s s.t. $s(x_0) = [u . v]$ //

principle

parallel section が多いほど

ホロノミー群は小さくなる

Proof

parallel transp w.r.t ∇

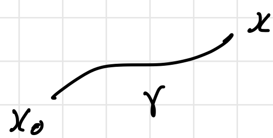
$$\nabla s = 0 \iff \forall \gamma \text{ curve}, \quad \Phi(\gamma)(s) = s$$

$\therefore \quad \gamma \in \Omega_{x_0}(M), \quad \Phi(\gamma)(s(x_0)) = s(x_0)$

$s(x_0) = [u_0, v]$ は $\rho(Hol(M))$-inv vector

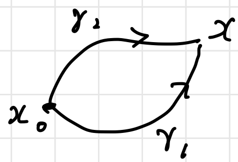
逆に, $s(x_0) = [u_0, v]$ が $\rho(Hol(M))$-inv とする.

$$s(x) := \Phi(\gamma)(s(x_0))$$



とすると well defined

$\because)$



$$\Phi(\gamma_2)^{-1} \circ \Phi(\gamma_1) = \Phi(\gamma_2^{-1}\gamma_1) \in \rho(Hol(M)) \,//$$

Ex (M, g) ∇ L-C $\rightsquigarrow \nabla g = 0$

$$\begin{array}{ccc} F(M) \curvearrowleft GL(n;\mathbb{R}) & & O(M) \curvearrowleft O(n) \\ \downarrow & \rightsquigarrow & \downarrow \\ M & & M \end{array}$$

inv vector $= \sum e_i \otimes e_i$ in $\mathbb{R}^n \otimes \mathbb{R}^n$

Ex (M^{2n}, g, J) Kähler mfd

$$\nabla g = 0, \quad \nabla J = 0$$

$$\begin{array}{ccc} O(M) & & U(M) \curvearrowleft U(n) \\ \downarrow & \rightsquigarrow & \downarrow \\ M & & M \end{array}$$

unitary frame o bundle

$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathrm{End}(\mathbb{R}^n)$ or $U(n)$-inv vector

Ex $X \in \mathfrak{X}(M)$ $\quad \nabla X = 0$

$$\mathrm{Hol}(M) \subset O(n-1)$$

Ex ω n-form $\quad \nabla\omega = 0$

$$\Rightarrow \mathrm{Hol}(M) \subset SO(n)$$

$\mathrm{Hol}(M)$ に関するいくつかの Fact

- $(\tilde{M}, \tilde{g})$ universal covering of (M, g)

  $\pi : \tilde{M} \to M$ covering, locally isometric

  $\pi_1(M) = \{1\}$

  $$\mathrm{Hol}(\tilde{M}, \tilde{g}) \cong \mathrm{Hol}^0(M, g)$$

  $\rightsquigarrow$ M simply conn として考えよう

- $(M, g) = (M_1, g_1) \times (M_2, g_2)$ Riem prod

  $$\Rightarrow \mathrm{Hol}(M) = \mathrm{Hol}(M_1) \times \mathrm{Hol}(M_2)$$

- (M^n, g) simply conn とする. $O(n) \curvearrowright \mathbb{R}^n$

$Hol(M) \curvearrowright \mathbb{R}^n$ irreducible (as rep)

$\Leftrightarrow$ (M, g) irreducible (as Riem mfd)

- ドラーム分解

(M, g) complete simp conn とする

$\Rightarrow$ $\exists (M_i, g_i)$ irr $i = 1, \cdots, k$

$$(M, g) = \prod_{i=1}^{k} (M_i, g_i)$$

$(Hol(M) = \prod Hol(M_i))$

- (M, g) sym space G/K $(\nabla R = 0)$

$Hol(M) \doteqdot K$

$\rightsquigarrow$ sym sp は Cartan により分類済み

$\mathbb{R}^n$ に irr に作用する $SO(n)$ の subgr を分類 $(\nabla R \neq 0)$

<u>Thm</u> (Berger)

(M^n, g) : complete, simp conn, irr
non-symmetric sp

$\Rightarrow$ $Hol(M)$ は次のいずれか.

(1) $Hol(M) = SO(n)$

(2) $n=2m$ $U(m)$

Ricci flat
(3) $n=2m$ $SU(m)$
(4) $n=4k$ $Sp(k)$

(5) $n=4k$ $Sp(k)Sp(1) = \dfrac{Sp(k)\times Sp(1)}{\{(1,1),(-1,-1)\}}$

Einstein

Ricci flat
(6) $n=7$ $G_2 \subset SO(7)$
(7) $n=8$ $Spin(7)^+ \subset SO(8)$

$\boxed{\text{各 Hol をもつ cpt mfd は，ちゃんと存在する!!}}$

(see Joyce の本)

§ geometric strs

Kähler geometry

M^{2n} mfd, $J \in \Gamma(M, \mathrm{End}(TM))$, $J^2 = -I$

$\rightsquigarrow$ $J_x : T_xM \otimes \mathbb{C} \to T_xM \otimes \mathbb{C}$ cpx linear

$T_xM^{1,0}$: $\sqrt{-1}$ eigenspace

$T_xM^{0,1}$: $-\sqrt{-1}$ eigenspace $\cong \overline{T_xM^{1,0}}$

$$T_xM \otimes \mathbb{C} \cong \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ (T_xM,\, J_x)}}{T_xM^{1,0}} \oplus \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ (T_xM,\, -J_x)}}{T_xM^{0,1}}$$

Def (1) (M, g, J) が almost Herm mfd とは,

とは, $g(JX, JY) = g(X, Y)$ $\forall X, Y$

($\rightsquigarrow$ $TM^{1,0}$ に Herm metric 入る)

(2) ∇ Levi-Civita on alm Herm mfd (M, g, J)

$\nabla J = 0$ のとき, (M, g, J) を

Kähler mfd という

また $\omega(X, Y) := g(JX, Y)$ 2-form

を Kähler form という.

$(\nabla \omega = 0)$

$\nabla J = 0$ より $\nabla TM^{0,1} \subset TM^{0,1}$

$X, Y \in \Gamma(M, TM^{0,1})$ に対して,

$[X, Y] = \nabla_X Y - \nabla_Y X \in \Gamma(M, TM^{0,1})$.

$\therefore$ $TM^{0,1}$ は可積分

$\rightsquigarrow$ (M, J) は cpx mfd,

つまり, $M = \bigcup_\alpha U_\alpha$, $\{(U_\alpha, \varphi_\alpha)\}$ atlas

$\varphi_\alpha : U_\alpha \longrightarrow \varphi_\alpha(U_\alpha) \subset \mathbb{C}^n = \mathbb{R}^{2n}$

cpx coordinate

s.t. $z_i = x_i + \sqrt{-1}\, y_i$

$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}$, $J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}$

とくに, $\varphi_\beta \circ \varphi_\alpha^{-1}$ は holomorphic //

<u>prop</u> (M, g, J) : kähler $\Leftrightarrow$ $Hol(M) \subset U(m)$

<u>proof</u> $J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ cpx str on $\mathbb{R}^{2m}$

$GL(m : \mathbb{C}) \cong \{ X \in GL(2m : \mathbb{R}) \mid J_0 X = X J_0 \} \subset GL(2m : \mathbb{R})$

$\cup$ $\cup$

$U(m) \ni X = A + \sqrt{-1}\, B \longmapsto \tilde{X} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in SO(2m)$

<u>Exc</u> · $X^* X = I_m \Leftrightarrow {}^t\tilde{X} \tilde{X} = I_{2m}$ $\left({}^\forall X \in GL(m : \mathbb{C}) \right)$

· $\det(\tilde{X}) = |\det X|^2 > 0$

$\sum e_i \otimes e_i$, J_0 は $U(m)$-inv vector $\rightarrow$ $\nabla g = 0$, $\nabla J = 0$

逆に，これらを保存するが $U(m)$ $\rightarrow$ $\Leftrightarrow$ $Hol(M) \subset U(m)$ //

(M, g, J): Kähler とする

$$T^*M \otimes \mathbb{C} \cong \Lambda^{1,0} \oplus \Lambda^{0,1}$$

$\Lambda^{1,0}$ の local frame $dz_1, \cdots, dz_m$

$\Lambda^{0,1}$ の local frame $d\bar{z}_1, \cdots, d\bar{z}_m$

$(\Lambda^{1,0} \cong TM^{0,1}, \ \Lambda^{0,1} \cong TM^{1,0})$

$$\Lambda^k T^*M \otimes \mathbb{C} \cong \bigoplus_{p+q=k} \underbrace{(\Lambda^{p,0} \otimes \Lambda^{0,q})}_{= \Lambda^{p,q}}$$

Ex

$$\Lambda^2 T^*M \cong \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}$$

$\Lambda^{1,1}$ の local frame $\{dz_i \wedge d\bar{z}_j\}$

$\Lambda^{1,1} \cong M \times \mathbb{C} \oplus \Lambda^{1,1}_0$ (irr decomp)

$\because)$ $\omega \in \Lambda^{1,1}$, $\nabla \omega = 0$

$M \times \mathbb{C} = \mathbb{C}\langle \omega \rangle$ //

$n = 2m$

さて $\Lambda^{m,0}$ の local frame $dz_1 \wedge \cdots \wedge dz_m$

$\det(\varphi_\beta \circ \varphi_\alpha^{-1})$ holo $\therefore$ holo line bundle

$K := \Lambda^{m,0}$ holo line bundle を

M上 canonical line bundle という

Def $\varphi = \sum_I \varphi_I(z)\, dz_I \in \Gamma(M, \Lambda^{k,0})$

φ_I holo fn のとき、φ holo k-form という

Calabi-Yau mfd

prop (M, g, J) Kähler mfd とする.

① $\exists \Omega$ nowhere vanishing holo m-form $\in \Gamma(M, K)$

$\Leftrightarrow$ ② K is holomorphically trivial.

$\Leftarrow$ ③ $\exists \Omega \in \Gamma(M, K)$ s.t. $\nabla \Omega = 0$

$\Leftrightarrow$ ④ $Hol(M) \subset SU(m)$

そして M cpt & Ricci flat なら ②$\Rightarrow$③

また③(or④)のとき (M, g, J, Ω) を Calabi-Yau mfd とよぶ, そして Ricci-flat であり

M の first Chern class $c_1(M) = 0$

注 Calabi-Yau mfd というときは, M: cpt と仮定することが多い

proof ③ $\Leftrightarrow$ ④

$U(m) \curvearrowright \mathbb{C}^m \rightsquigarrow U(m) \curvearrowright \Lambda^m(\mathbb{C}^m)$

1-dim cpx vector sp

$\det : U(m) \to U(1)$ で転られる

$\overline{\mathbb{C}^m} = (\mathbb{C}^m)^*$ as $U(m)$-module

$\Lambda^{m,0} \cong \Lambda^m(\overline{\mathbb{C}^m})$ $U(m)$ は $\overline{\det}$ で act

$K = U(M) \underset{\overline{\det}}{\times} \mathbb{C}$ $\overline{\det}$: trivial on $SU(m)$

④ とすると,

$v \neq 0 \in \Lambda^m(\overline{\mathbb{C}^m})$, $SU(m)$-inv vector

$\Omega \in \Gamma(M, K)$ s.t. $\nabla\Omega = 0$ をとれる $\therefore$ ③ o.k

③ $\Rightarrow$ ④ 同様 (④ $\Rightarrow$ Ricci flat あとで)

① $\Leftrightarrow$ ② も easy.

② $\Rightarrow$ ③ は cpt & $Ric = 0$ が必要.

kähler mfd 上 $\Omega^{p,q} := \Gamma(M, \Lambda^{p,q})$

diff op $\bar\partial : \Omega^{p,q} \longrightarrow \Omega^{p,q+1}$, $\bar\partial^* : \Omega^{p,q} \longrightarrow \Omega^{p,q-1}$ があり

BW公式 $\bar\partial \circ \bar\partial^* + \bar\partial^* \circ \bar\partial = \nabla^*\nabla +$ 曲率項

$K = \Lambda^{m,0}$ 上で 曲率項 $= Ric$

∴ Ricci flat なら,

$$\bar\partial^*\bar\partial = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial = \nabla^*\nabla \quad \text{on } \Omega^{m,0}$$

$$\left(\bar\partial^* = 0 \ \text{on } \Omega^{m,0} \right)$$

Ω holo とすると $\bar\partial\Omega = 0$

∴ cpt Ricci flat kähler mfd 上で

$0 = \int_M |\nabla\Omega| \, vol_g$ となり, $\nabla\Omega = 0$ ∴ ② ⇒ ③

<u>Note</u> Fact

(M, J) cpt cpx mfd, K is holo trivial

⇒ $\exists g$ metric $Hol(M) \subset SU(m)$

(by <u>Calabi 予想の解決</u>)

<u>Spinors on kähler mfd</u>

$\pi_1(SU(m)) = 1$, i.e. $SU(m)$ simply conn.

$$\begin{tikzcd}
 & Spin(2m) \arrow[d] \\
SU(m) \arrow[r, hook, "\iota"'] \arrow[ur, "\exists^1 \text{lift } \tilde\iota"] & SO(2m)
\end{tikzcd}$$

$Hol(M) \subset SU(m)$ なら $\exists$ canonical spin str

$$Spin(M) = SU(M) \times_{\tilde\iota} Spin(n)$$

Spinor bundle は、

$$S = Spin(M) \times_{\Delta} W_n$$
$$= SU(M) \times_{\Delta \circ \tilde{\iota}} W_n$$ となる

しかし，$SU(m) \curvearrowright W_n$ は irr でない

<u>Fact</u>

$$W_n \cong \bigoplus_{p=0}^{m} \Lambda^p(\mathbb{C}^m)$$

$$W_n^+ \cong \bigoplus_{l=0} \Lambda^{2l}(\mathbb{C}^m) \quad W_n^- \cong \bigoplus_{l=0} \Lambda^{2l+1}(\mathbb{C}^m) \,/\!/$$

$\therefore$ $S \cong \bigoplus_{p=0}^{m} \Lambda^{0,p}$ となる

そして $p=0.m$ は $SU(m)$ の自明表現なので，

$\Lambda^{0.0} \oplus \Lambda^{0.m} \subset S$ は

自明 asso vector bundle （holo bundle としても自明）

$\therefore$ 2つの独立な parallel spinor が存在

$\therefore$ $Hol(M) \subset SU(m) \Rightarrow$ Ricci flat

<u>Note</u>

$$\Lambda^{0.0} \subset S^+, \quad \Lambda^{0.m} \subset \begin{cases} S^+ & m = \text{even} \\ S^- & m = \text{odd} \end{cases}$$

Note (M, g, J) almost hermitian mfd

$$w_2(M) = c_1(K) \mod 2$$
$$= -c_1(M)$$

$\therefore$ $c_1(M) \equiv 0 \mod 2$ なら spin str が存在

このとき $c_1(K)/2 \in H^2(M, \mathbb{Z})$

$\therefore$ L cpx line bundle s.t. $L^2 = K$
$\quad$ $\sqrt{K}$ とかくが、存在しても1つとは限らない。

そして $S \cong \bigoplus_{p=0}^{m} \Lambda^{0,p} \otimes \sqrt{K}$

なお、cpt kähler mfd なら、

spin str $\overset{1:1}{\longleftrightarrow}$ $\{L \mid L \to M \text{ holo line}, L^2 = K\}$

Note kähler なら canonical spin^c-str をもつ

$$S' = \bigoplus_{p=0}^{m} \Lambda^{0,p} \qquad D' \text{ spin}^c\text{-Dirac} = \bar{\partial}^* + \bar{\partial} \quad //$$

hyper kähler mfd

Def (M^{4k}, g) が hyper kähler mfd とは、

$\exists I_1, \exists I_2 \in \Gamma(M, \mathrm{End}(TM))$

- $I_3 := I_1 I_2 = -I_2 I_1$, • $I_i^2 = -1$ • $\nabla I_i = 0$
- $g(I_i X, I_i Y) = g(X, Y)$ $\quad i = 1, 2, 3$

モデルは $\mathbb{H}^k \cong \mathbb{R}^{4k}$

$p \in \mathbb{H}$ を右から $x \mapsto x\bar{p}$

とすると $\mathbb{H}^k$ は $\mathbb{H}$-linear space

$GL(k, \mathbb{H}) \curvearrowright \mathbb{H}^k$ $\quad x \mapsto Ax$ (左)

は $\mathbb{H}$-linear

内積は, $\langle x, y \rangle = Re(x\bar{y}) = \sum Re(x_i \bar{y}_i)$

$I_1(x) = x\bar{i}$, $I_2(x) = x\bar{j}$

$\langle I_1(x), I_1(y) \rangle = Re(x\bar{i}\,\overline{y\bar{i}})$

$= Re(x\bar{i}\,i\bar{y}) = Re(x.\bar{y})$

$\therefore$ I_1, I_2, I_3, g_0 on $\mathbb{H}^k$ は,

$Sp(k) = GL(k : \mathbb{H}) \cap SO(4k)$ -invariant

(逆も o.k.)

そこで (M^{4k}, g) が hyper kähler mfd

$\Leftrightarrow$ $Hol(M) \subset Sp(k) \subset SO(4k)$

とくに, $Sp(k) \subset SU(2k)$ より

hyper kähler $\Rightarrow$ spin mfd,

Ricci flat.

Spinors on h-k-mfd

(M^{4k}, g, I_1, I_2) h-k mfd

$\rightsquigarrow (M^{4k}, g, I_1)$ kähler $\quad Hol(M) \subset SU(2k)$

$$S \cong \bigoplus_{p=0}^{2k} \Lambda^{0,p}$$

irr かをみる

$$\omega(X, Y) := g(I_2 X, Y) + \sqrt{-1}\, g(I_3 X, Y)$$

は 2-form であるが

・$\omega(I_1 X, Y) = \sqrt{-1}\, \omega(X, Y) = \omega(X, I_1 Y)$

・$\nabla \omega = 0$ とくに holo w.r.t. I_1,

$\therefore$ ω: (parallel) nondeg holo 2-form $\in \Gamma(M, \Lambda^{2,0})$

$\therefore$ ω^l : parallel $2l$-form

$l = 1, 2, \cdots, k$.

また ω holo nondeg より

$\omega : TM^{1,0} \times TM^{1,0} \longrightarrow \mathbb{C}$ nondeg

$$\Lambda^{1,0} \cong (TM^{1,0})^* \underset{\omega}{\cong} TM^{1,0} \underset{\substack{\uparrow \\ (M, g, I_1)\ \text{ケーラー}}}{\cong} \Lambda^{0,1}$$

$$S \cong \bigoplus \Lambda^{0,p} \cong \bigoplus \Lambda^{p,0}$$

$$\Lambda^{2l,0} \cong \mathbb{C}\langle \omega^l \rangle \oplus \Lambda_0^{2l,0}$$

$$l = 0, 1, \cdots, k$$

$\therefore S$ on n-k mfd は

$k+1$ 個の parallel spinor

$\{\omega^l \mid l = 0, 1, 2, \cdots, k\}$ をもつ

G_2-mfd

$\boxed{\text{octonion}}$

On $\mathbb{H} \oplus \mathbb{H}$,

$$(a, b)(c, d) := (ac - \bar{d}b, da + b\bar{c})$$

により、non comm, nonasso ($(xy)z \neq x(yz)$)

な $\mathbb{R}$ 上 alg $\mathbb{O}$ を得る ($\dim_{\mathbb{R}} \mathbb{O} = 8$) //

- $\mathbb{O}$ の 共役 $\overline{(a, b)} := (\bar{a}, -b)$
- $\langle x, y \rangle := Re(x\bar{y})$
- $Re(\mathbb{O}) = \{(a, 0) \in \mathbb{H} \oplus \mathbb{H} \mid a \in \mathbb{R}\} \cong \mathbb{R}$
- $Im(\mathbb{O}) = (Re\,\mathbb{O})^{\perp} \cong \mathbb{R}^7$
- $\|x\|\|y\| = \|xy\|$, $\quad x^{-1} = \dfrac{\bar{x}}{\|x\|^2} \quad (x \neq 0)$

• クロス積 on $\mathbb{O}$

$$x \times y := \mathrm{Im}(\bar{y}x) = \frac{1}{2}(\bar{y}x - x\bar{y})$$

bilinear & anticommute

さらに、 $x, y \in \mathrm{Im}(\mathbb{O})$ に対して

$\langle x, y\rangle = -\frac{1}{6}\mathrm{tr}(x \times (y \times))$ となる。☆

(クロス積 ⇝ 内積)

• 3重クロス積 on $\mathbb{O}$,

$$x \times y \times z := \frac{1}{2}(x(\bar{y}z) - z(\bar{y}x)) \in \mathbb{O}$$

trilinear & anticommute

さらに、 $x, y, z \in \mathrm{Im}(\mathbb{O})$ に対して

$\mathrm{Re}(x \times y \times z) = \langle x, yz\rangle$ となる。

Def

(1) $\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$ 上 associative 3-form

$$\phi(x, y, z) := \langle x, yz\rangle$$

(2) $\mathbb{O} \cong \mathbb{R}^8$ 上の Cayley 4-form

$$\Phi(x, y, z, w) := \langle x, y \times z \times w\rangle$$

<u>def</u> $\mathbb{O} \cong \mathbb{R}^8$ を考える

$$G_2 := \{ g \in GL(8;\mathbb{R}) \mid g(x)g(y) = g(xy) \ \forall x,y \in \mathbb{O} \}$$

($\pi_1(G_2) = 1$, $\dim G_2 = 14$ の Lie gr)

$1 = (1,0) \in \mathbb{O}$ に対し

・$g(1)g(1) = g(1^2) = g(1)$, $\|g(1)\| = 1$

∴ $g(1)^{-1}$ 存在 $g(1) = 1$

$g(\mathrm{Re}\,\mathbb{O}) = \mathrm{Re}(\mathbb{O})$ ∴ $G_2 \subset GL(\mathrm{Im}\,\mathbb{O})$

G_2 は $\mathbb{O}$ の積を保つ → クロス積を保つ

∴ ★ より $\mathrm{Im}(\mathbb{O})$ の内積を保つ

また $\mathrm{vol} = \phi \wedge * \phi$ on $\mathrm{Im}(\mathbb{O})$ も保つ

以上より $G_2 \subset SO(7)$

$\pi_1(G_2) = 1$ より

$$\begin{tikzcd} & Spin(7) \arrow[d] \\ G_2 \arrow[ur, "\exists!\ \tilde{\iota}"] \arrow[r, hook, "\iota"'] & SO(7) \end{tikzcd}$$

canonical lift $\tilde{\iota}$ がいる.

<u>Fact</u> $G_2 = \{ g \in GL(\mathrm{Im}\,\mathbb{O}) \mid g^*\phi = \phi \}$

また, W_7 (cpx 8-dim) $\cong \mathbb{C} \oplus \mathbb{C}^7$

as G_2-module

$\mathbb{C}$: trivial rep

$\mathbb{C}^7$: natural rep of $G_2 \subset SO(7)$

<u>Def</u> (M, g) 7-dim Riem mfd

$\mathrm{Hol}(M) \subset G_2 \ (\subset SO(7))$

のとき G_2-mfd という。

G_2-inv 3-form $\phi \rightsquigarrow \nabla\phi = 0$

<u>Prop</u> (M, g) : G_2-mfd

$\Rightarrow$ (1) $\exists \phi$ associative 3-form $\nabla\phi = 0$

(2) canonical spin str をもつ

$S \cong M \times \mathbb{C} \oplus TM \otimes \mathbb{C}$

とくに $\exists \varphi$ parallel spinor

(3) Ricci flat。

<u>Note</u> $M^7 + \phi$ associative 3-form

$$\rightsquigarrow \begin{array}{c} G_2(M) \curvearrowleft G_2 \\ \downarrow \\ M \end{array} \quad G_2 \subset SO(7) \quad \rightsquigarrow g \text{ Riem}$$

almost G_2-mfd

(M^7, φ) almost G_2 とする

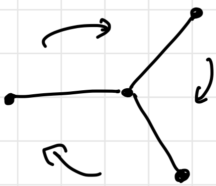
$\mathrm{Hol}(M) \subset G_2 \iff \nabla\varphi = 0$

$\iff d\varphi = 0$ & $d*\varphi = 0$

∇ は φ から定まる ($\because$ φ に対する nonlinear eq) //

Spin(7)-mfd

$\mathrm{Spin}(7) \subset SO(8)$, て何!?



Dynkin diag for $SO(8)$

$\rightsquigarrow$ 外部自己同型 あり

$\rightsquigarrow$ $\mathrm{Spin}(8)$ の irr 8-dim (real) rep は3つ

$\mathrm{Ad} : \mathrm{Spin}(8) \to SO(8)$ $\quad \mathrm{Spin}(8) \overset{\mathrm{Ad}}{\curvearrowright} \mathbb{O}$

$\underset{\sim}{\Delta_8^{\pm}} : \mathrm{Spin}(8) \curvearrowright \mathbb{O} \ (= \mathbb{O}^{\pm})$

← これをつくる

$$\mathbb{R}^8 \cong \mathbb{O} \ni x \longmapsto A_x = \begin{pmatrix} 0 & L_x \\ -L_{\bar{x}} & 0 \end{pmatrix} \text{ on } \underset{\mathbb{O}^+}{\underset{\shortparallel}{\mathbb{O}}} \oplus \underset{\mathbb{O}^-}{\underset{\shortparallel}{\mathbb{O}}}$$

ここで $L_x(y) = xy$ in $\mathbb{O}$

このとき $A_x A_y + A_y A_x = -2\langle x, y\rangle$ となる

$$\therefore \quad Cl_8 \cong \mathbb{R}(16) \cong \mathrm{End}_{\mathbb{R}}(\mathbb{O}^+ \oplus \mathbb{O}^-)$$

$$\left(\begin{array}{l} Ae_1 Ae_2 \cdots Ae_7 = \begin{pmatrix} I_8 & 0 \\ 0 & -I_8 \end{pmatrix} \\[2ex] \mathbb{C}l_8 = Cl_8 \otimes \mathbb{C} \curvearrowright \underbrace{\mathbb{O}^+ \otimes \mathbb{C}}_{= W_8^+} \oplus \underbrace{\mathbb{O}^- \otimes \mathbb{C}}_{= W_8^-} \end{array} \right.$$

また. $\mathbb{R}^7 \hookrightarrow \mathbb{R}^8 \quad (\mathrm{Im}\,\mathbb{O} \subset \mathbb{O})$

$$\mathrm{Spin}(7) = \{ g = v_1 \cdots v_{2p} \mid v_i \in \mathbb{R}^7,\ p = 1, 2, \cdots \}$$
$$\subset \mathrm{Spin}(8)$$

これは $\mathrm{Spin}(7) = \{ g \in \mathrm{Spin}(8) \mid Ad(g) e_0 = e_0 \}$

$e_0 \in \mathrm{Re}(\mathbb{O}) \subset \mathbb{O}$

同様に

$$\mathrm{Spin}(7)^{\pm} = \{ g \in \mathrm{Spin}(8) \mid \Delta_8^{\pm}(g) e_0 = e_0 \}$$
$$\cong \mathrm{Spin}(7) \qquad (e_0 \in \mathbb{O}^{\pm})$$

$$\begin{tikzcd}
 & & \mathrm{Spin}(7)^+ \arrow[dd, bend left] \\
\mathrm{Spin}(7) \arrow[urr, bend left] \arrow[r, phantom, "\subset"] & \mathrm{Spin}(8) \arrow[ur, phantom, "\supset"] \arrow[dr, phantom, "\supset"] & \\
 & & \mathrm{Spin}(7)^- \arrow[ull, bend left]
\end{tikzcd}$$

$Ad : \mathrm{Spin}(8) \to SO(8)$ で

$Ad|_{Spin(7)}(\pm 1) = 1$ not inj

しかし $Ad|_{Spin(7)^+}$ は inj

$\therefore \quad Spin(7)^+ \subset SO(8) \overset{irr}{\curvearrowright} \mathbb{O}$

<u>Fact</u> $\mathbb{O}$ 上の Cayley 4-form Φ

$$Spin(7)^+ = \{ g \in GL(8:\mathbb{R}) \mid g^*\Phi = \Phi \} \quad //$$

さて,

$$\begin{tikzcd} & Spin(8) \arrow[d] \\ Spin(7)^+ \arrow[ur, "\tilde{\iota}"] \arrow[r, hook, "\iota"] & SO(8) \end{tikzcd}$$

canonical lift がある

また $Spin(8) \curvearrowright \mathbb{O}^+ \otimes \mathbb{C} \cong W_8^+$

$e_0 \in \mathbb{O}^+ \otimes \mathbb{C}$ は $Spin(7)^+$-inv vector

<u>Note</u> $Spin(7)^+ \curvearrowright \mathbb{O} \cong \mathbb{R}^8$

$(1,0) = e_0 \in \mathbb{O}$, $\quad G_{e_0} \cong G_2$

$\therefore \quad Spin(7)^+ / G_{e_0} \cong S^7 \subset \mathbb{O}$

同様に $\quad Spin(7) \curvearrowright \mathbb{O}^+$

$$e_0 \in \mathbb{O}^+, \quad \begin{tikzcd} G_{e_0} = G_2 \arrow[r, hook] \arrow[dr] & Spin(7) \arrow[d, "Ad"] \\ & SO(7) \end{tikzcd}$$

$e_0 \in \mathbb{O}^+ \otimes \mathbb{C} \cong W_7$ が G_2-inv vector $\quad (W_7 \cong \mathbb{C} \oplus \mathbb{C}^7) \quad //$

Def (M, g) 8-dim Riem mfd

$\mathrm{Hol}(M) \subset \mathrm{Spin}(7)^{+} \subset SO(8)$ なとき

Spin(7)-mfd という

prop (M, g) Spin(7)-mfd $(\Phi = *\Phi)$

$\Rightarrow$ (1) Φ Cayley 4-form $\nabla \Phi = 0$

(2) canonical spin str をもち

$$S^{+} \cong \underline{M \times \mathbb{C}} \oplus \Lambda^2_7(M)$$

$$S^{-} \cong TM \otimes \mathbb{C}$$

$\Lambda^2(M)$ の 7-dim irr comp

(3) $\exists \varphi$ parallel spinor

(4) Ricci flat mfd

Note M, Φ almost Spin(7)-mfd

$\mathrm{Hol}(M) \hookrightarrow \mathrm{Spin}(7) \Leftrightarrow \nabla \Phi = 0$

$\Leftrightarrow d\Phi = 0$ //

§ 分類定理

これまでの考察から

(M, g) complete, simply connected

irreducible (as Riem), Spin mfd

with φ : parallel spinor

$\therefore$ (M, g) Ricci flat

Step 1 (M, g) sym sp とすると

irr, simply conn, Ricci flat より

$(M, g) \cong (\mathbb{R}^n, g_E)$

not irr

Step 2 (M, g) non sym sp

Bergerの分類より,

$Hol(M) = SO(n), U(m), SU(m), Sp(k)$

$Sp(k)Sp(1), Spin(7), G_2$

- parallel spinor をもつ $\cancel{SO(n)}$
- $Sp(k)Sp(1) \Rightarrow$ Einstein mfd

  $Scal = 0$ より, $Hol(M) = Sp(k)$
- $U(m) \Rightarrow$ Ricci flat kähler

  $\therefore$ $Hol(M) = SU(m)$

$SU(m)$, $Sp(k)$, G_2, $Spin(7)$ - mfd

parallel spinor をもった

<u>Theorem</u> (Wang, 1989)

(M^n, g) : complete, simply conn, irr, spin mfd $(\dim M \geqq 2)$

$N = \{ \varphi \in \Gamma(M, S) \mid \nabla \varphi = 0 \}$

$N^{\pm} = \{ \varphi \in \Gamma(M, S^{\pm}) \mid \nabla \varphi = 0 \}$ とする

$\dim N > 0$ なら, (M, g) は次のいずれか

$\dim M$	$Hol(M)$	geom str	$(\dim N^+, \dim N^-)$
$4k$	$SU(2k)$	カラビ-ヤウ	$(2, 0)$
$4k+2$	$SU(2k+1)$	カラビヤウ	$(1, 1)$
$4k$	$Sp(k)$	超ケーラー	$(k+1, 0)$
7	G_2	G_2	1
8	$Spin(7)$	$Spin(7)$	$(1, 0)$

次に real killing spinor をもつ mfds

(Friedrich estimate の limiting mfd)

を分類する.

Riemannian cone (リーマン錐)

Def $(M.g)$ n-dim Riem mfd

$$\bar{M} := M \times \underset{(0,\infty)}{\mathbb{R}^+}, \quad \bar{g} = r^2 g + dr^2$$

$$\pi : \bar{M} \ni (x, r) \longmapsto x \in M$$

この $(\bar{M}.\bar{g})$ を $(M.g)$ の リーマン錐 という//

$\bar{\nabla}$: $\bar{g}$ の Levi-Civita conn

Exc Koszul公式をつかって、

$$\begin{cases} \bar{\nabla}_{\partial r}\partial r = 0, \quad \bar{\nabla}_{\partial r} X = \dfrac{X}{r} = \bar{\nabla}_X \partial r \\ \bar{\nabla}_X Y = \nabla_X Y - r\, g(X, Y)\partial r \qquad \left(\partial r = \dfrac{\partial}{\partial r}\right) \end{cases}$$

を示せ。ここで $X \in \mathfrak{X}(M)$ を

$$X_{(x,r)} := (X_x, 0) \in T_x M \oplus T_r \mathbb{R}^+$$

により $X \in \mathfrak{X}(\bar{M})$ へ extendしている。

Exc $(\bar{M}.\bar{g})$ の曲率 $\bar{R}$ は、

- $\bar{R}(X, \partial r)\partial r = \bar{R}(X.Y)\partial r = \bar{R}(X.\partial r)Y = 0$
- $\bar{R}(X.Y)Z = R(X.Y)Z + g(X.Z)Y - g(Y,Z)X$

となることを示せ

とくに $\bar{M}$ flat $\Longleftrightarrow$ M は S^n に isometric //

このExcより Ricci 曲率とスカラー曲率は,

$$\begin{cases} \overline{Ric}(X,Y) = Ric(X,Y) - (n-1)g(X,Y) \\ \overline{Scal}(x,r) = \frac{1}{r^2}(Scal(x) - n(n-1)) \end{cases}$$

<u>目標</u> (M,g) 上の killing Spinor

$\downarrow$

$(\overline{M},\overline{g})$ 上の parallel spinor "

$SO(M)$: (M,g) の oriented o.n. fram bundle

$$SO_{n+1}(M) := SO(M) \times_i SO(n+1)$$

$$i : SO(n) \ni g \longmapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \in SO(n+1) \circlearrowright \mathbb{R}^{n+1}$$

$$SO_{n+1}(M) \times_i \mathbb{R}^{n+1} \cong TM \oplus M \times \mathbb{R}$$

canonical section of $M \times \mathbb{R}$ を

$e_{n+1}(x) = (x, 1)$ とかく

このとき $\pi^* SO_{n+1}(M) \cong SO(\overline{M})$ on $\overline{M}$

$\because)$ $\pi^* SO_{n+1}(M) \ni (e_1, e_2, \cdots, e_n, e_{n+1})$ ✳

$\mapsto (\overline{e_1}, \cdots, \overline{e_n}, \overline{e_{n+1}} = \partial r) \in SO(\overline{M})$

ここで $\overline{e_i} = e_i / r$ $\square$

同様に

$$\mathrm{Spin}_{n+1}(M) := \mathrm{Spin}(M) \times_{\iota} \mathrm{Spin}(n+1)$$

とすると

$$\pi^* \mathrm{Spin}_{n+1}(M) \cong \mathrm{Spin}(\bar{M})$$

次に spinor space の対応をみる

<u>$n=2m$</u>

$$\begin{array}{ccccccc} \mathrm{Spin}(2m) & \subset & \mathbb{C}l_{2m} \cong \mathbb{C}(2^m) & \curvearrowright & \mathbb{C}^{2^m} = W_{2m} & & \\ \cap & & \wr\wr & \longleftarrow & e_i \longmapsto e_i e_{2m+1} & & i=1,\cdots,2m \\ \mathrm{Spin}(2m+1) & \subset & \mathbb{C}l^0_{2m+1} \cong \mathbb{C}(2^m) & & (**) & & \end{array}$$

$$\Delta_{2m+1} : \mathrm{Spin}(2m+1) \curvearrowright \mathbb{C}^{2^m} = W_{2m+1}$$

を Spin(2m) へ restrict

$$\begin{cases} \Delta_{2m+1}\big|_{\mathrm{Spin}(2m)} \cong \Delta_{2m} \\ W_{2m+1} \cong W_{2m} \end{cases}$$

<u>$n=2m-1$</u>

$$\begin{array}{ccc} \mathrm{Spin}(2m-1) \subset \mathbb{C}l_{2m-1} & \cong \mathbb{C}(2^{m-1}) \oplus \mathbb{C}(2^{m-1}) & \overset{\rho_\pm}{\curvearrowright} \mathbb{C}^{2^{m-1}} \\ \wr\wr & & \\ \mathrm{Spin}(2m) \subset \mathbb{C}l^0_{2m} & & \end{array}$$

$$\Delta^{\pm}_{2m} = \rho_\pm\big|_{\mathrm{Spin}(2m)}$$

$$\Delta_{2m-1} \cong \rho_\pm\big|_{\mathrm{Spin}(2m-1)}$$

$$\therefore \quad \Delta^{\pm}_{2m}\big|_{\mathrm{Spin}(2m-1)} \cong \Delta_{2m-1}, \quad W^{\pm}_{2m} \cong W_{2m-1} \,_{//}$$

以上から、 $\mathrm{Spin}(n+1) \subset Cl_n \cong Cl^0_{n+1}$

$\cup \qquad\qquad \cup$

$\mathrm{Spin}(n) \subset Cl^0_n$

この埋め込みは $SO(n) \hookrightarrow SO(n+1)$ から導かれてるものと同じ

さて. $n = 2m$

$$Cl_{2m} = \mathbb{C}(2^m) \curvearrowright \mathbb{C}^{2^m} = W_{2m}$$

$\cong$ (縦)

Cl^0_{2m+1}

この Cl_{2m} の rep を $\mathrm{Spin}(2m+1)$ へ restrict すると

$(\Delta_{2m+1}, W_{2m+1})$ と同値である。

一方 $\mathrm{Spin}(2m)$ へ restrict したものが

(Δ_{2m}, W_{2m}) である.

$$\therefore \begin{cases} \Delta_{2m+1}\big|_{\mathrm{Spin}(2m)} \cong \Delta_{2m} \\ W_{2m+1} \cong W_{2m} \quad (\text{as } \mathrm{Spin}(2m) \text{ rep space}) \end{cases}$$

$n = 2m-1$

$$\begin{array}{ccccc} & & Cl^+_{2m-1} & & Cl^-_{2m-1} \\ & & \cong & & \cong \\ Cl_{2m-1} & \cong & \mathbb{C}(2^{m-1}) & \oplus & \mathbb{C}(2^{m-1}) \curvearrowright \mathbb{C}^{2^{m-1}} \\ \cong & & & & \\ Cl^0_{2m} & & & & \end{array}$$

以上より

$$S = Spin(M) \times_{\Delta_n} W_n$$

$$\cong \begin{cases} Spin_{n+1}(M) \times_{\Delta_{n+1}} W_{n+1} & (n=2m) \\ Spin_{n+1}(M) \times_{\Delta_{n+1}} W_{n+1}^{\pm} & (n=2m-1) \end{cases}$$

$$\bar{S} = Spin(\bar{M}) \times_{\Delta_{n+1}} W_{n+1}, \quad \bar{S}^{\pm}$$

は

$$\pi^* S \cong \begin{cases} \bar{S} & (n=2m) \\ \bar{S}^{\pm} & (n=2m-1) \end{cases}$$

そこで

$$\varphi \in \Gamma(M, S) \rightsquigarrow \pi^* \varphi \in \Gamma(\bar{M}, \bar{S} \text{ or } \bar{S}^{\pm})$$

(*), (**) より,

$$\begin{cases} \underbrace{\frac{1}{r} X}_{\bar{e}_i} \cdot \underbrace{\partial_r}_{\bar{e}_{n+1}} \cdot \pi^* \varphi = \pi^* (\underbrace{X}_{e_i} \cdot \varphi) \\ \frac{1}{r} X \cdot \frac{1}{r} Y \cdot \pi^* \varphi = \pi^* (X \cdot Y \cdot \varphi) \end{cases}$$

また $\bar{\nabla} = d + \frac{1}{2} \sum \bar{g}(\bar{\nabla} \bar{e}_i, \bar{e}_j) \bar{e}_i \cdot \bar{e}_j \cdot$

と $\bar{\nabla}$ の Exc より

$$\bar{\nabla}_{\partial r}\bar{\varphi} = \frac{\partial \bar{\varphi}}{\partial r}, \quad \bar{\nabla}_{e_k}\bar{\varphi} = \nabla_{e_k}\bar{\varphi} - \frac{1}{2}\bar{e}_k\cdot\partial r\cdot\bar{\varphi}$$
$$(\forall \bar{\varphi} \in \Gamma(\bar{M}, \bar{S}))$$

∴ $\varphi \in \Gamma(M, S)$ に対して

$$\begin{cases} \bar{\nabla}_{\partial r}\pi^*\varphi = 0, \\ \bar{\nabla}_X \pi^*\varphi = \pi^*\left(\nabla_X\varphi - \frac{1}{2}X\cdot\varphi\right) \end{cases}$$

とくに φ : killing spinor on (M, g)

$\Rightarrow$ $\pi^*\varphi$: parallel spinor on $(\bar{M}, \bar{g})$

逆に, $\bar{\varphi}$ parallel spinor なら

$$\bar{\nabla}_{\partial r}\bar{\varphi} = \frac{\partial\bar{\varphi}}{\partial r} = 0 \quad (r\text{方向に const})$$

$$\therefore\ \bar{\varphi} = \pi^*(\bar{\varphi}|_{\underbrace{M}_{(r=1)}})$$

$$0 = \bar{\nabla}_{e_k}\bar{\varphi} = \nabla_{e_k}\bar{\varphi} - \frac{1}{2}\bar{e}_k\cdot\partial r\cdot\bar{\varphi} \text{ より}$$

$\bar{\varphi}|_M$ は killing spinor.

$n = 2m-1$ のとき, $\pi^* S \cong \bar{S}^-$ をもちいると

$$\bar{\nabla}_X\pi^*\varphi = \pi^*\left(\nabla_X\varphi + \frac{1}{2}X\cdot\varphi\right) \text{ となる.}$$

<u>Thm</u> (M, g) spin mfd $\quad (\bar{M}, \bar{g})$ Riem cone

M 上 real killing spinor on S

$\updownarrow$ 1:1

$\bar{M}$ 上 parallel spinor on $\bar{S}$ (or $\bar{S}^{\pm}$)

ここで $\bar{M}$ へ Wang の result を apply しよう

<u>Fact</u> (Gallot 1979)

(M, g) complete Riem mfd

$\Rightarrow (\bar{M}, \bar{g})$ irr or flat

とくに $\Leftarrow$ M simply conn, cpt, $Hol(\bar{M})$ red

ならば $(\bar{M}, \bar{g})$ flat i.e. (M, g) Sphere.

(M, g) : simply conn, complete, spin

with real killing spinor φ

(killing number $\pm\frac{1}{2}$)

$\rightsquigarrow$ · cpt Einstein, irreducible

$Scal = n(n-1)$

· $\overline{Ric} = 0$. $\quad \pi^{*}\varphi$ parallel

$\bar{M}$: reducible なら, M は S^n

$\bar{M}$: irr なら $\bar{M}$ の Ricci $= 0$ (∵ nonsym)

Wang の result より

$Hol(\bar{M}) = SU(m), Sp(k), Spin(7), G_2$

$K_\pm := \{ \varphi \in \Gamma(M, S) \mid \nabla_X \varphi = \pm \frac{1}{2} X \cdot \varphi \}$

<u>$n = 2m$</u>

$\dim \bar{M} = 2m+1$

with parallel spinor

∴ $Hol(\bar{M}) = G_2$ ($\dim \bar{N} = 1$)

$\exists \varphi$ killing spinor ($k.n = \frac{1}{2}$)

$\varphi = \varphi_+ + \varphi_- \longrightarrow \varphi_+ - \varphi_-$ $k.n. = -\frac{1}{2}$

∴ $\dim K_\pm = 1$

<u>Fact</u>

(M^6, g) N.K str $\Leftrightarrow (\bar{M}, \bar{g})$ G_2-str

<u>outline</u>

ϕ ass 3-form on G_2 on $\bar{M}$

⇝ $g(X, JY) = \phi(\partial r, X, Y)$ on M

によりJを def

(M^6, g, J) almost Herm

また,

$$g((\nabla_X J)(Y), (\nabla_X J)(Y))$$

$$= g(X,X)g(Y,Y) - g(X,Y)^2 - g(JX,Y)^2$$

$\therefore$ $Y = X$ のとき $\underline{(\nabla_X J)(X) = 0}$

but $\nabla_X J \neq 0$

$\therefore$ (M^6, g) Nearly Kähler mfd $_{//}$

このとき (M^6, g, J) N-K

$$\phi := r^2 dr \wedge \omega + \frac{1}{3} r^3 d\omega \text{ on } \bar{M}$$

は associative 3-form & $\bar{\nabla}\phi = 0$

$\therefore$ $(\bar{M}, \bar{g})$ G_2-mfd $_{//}$

$\underline{n = 2m-1}$

$$Hol(\bar{M}) = SU(2k) \quad (n+1 = 4k)$$

$$\dim N_+ = 2, \quad \dim N_- = 2$$

on M $\dim K_+ = 2$, $\dim K_- = 2$

$\bar{M}$ Calabi-Yau $\underset{1:1}{\longleftrightarrow}$ M Ein Sasaki

Def (M, g) with ξ vector field,

が Sasaki mfd とは

(1) ξ killing vector field

(2) $\phi := -\nabla\xi$ (1.1) tensor

$\eta := g(\xi, \cdot)$ 1-form

とおくと

$$\phi\circ\phi = -id + \eta\otimes\xi$$

(3) $(\nabla_X\phi)(Y) = g(X, Y)\xi - \eta(Y)X$ //

$\bar{M}$: ケーラー mfd

$\xi := J(r\partial r)$ により M Sasaki

$\bar{M}$: Calabi Yau $\Rightarrow$ M : Einstein

逆に

$$\begin{cases} J(r\partial r) = \xi, \quad J(\xi) = -r\partial r \\ J(X) = -\phi(X) \end{cases}$$

により M Einstein Sasaki

$\Rightarrow \bar{M}$ Calabi Yau. //

Thm Bär (1993)

(M^n, g) complete simply conn, spin mfd with real killing spinor.

$\Rightarrow$ (M, g) は次のいずれか.

$n = \dim M$	$Hol(\bar{M})$	geom str	$\dim K^+, K^-$
n	$\{1\}$	S^n	$(2^{[\frac{n}{2}]}, 2^{[\frac{n}{2}]})$
$4k-1$	$SU(2k)$	Sasaki-Ein	$(2, 0)$
$4k+1$	$SU(2k+1)$	Sasaki-Ein	$(1, 1)$
$4k-1$	$Sp(k)$	3-Sasaki	$(1, 1)$
6	G_2	nearly kähler	$(1, 0)$
7	$Spin7$	nearly parallel G_2	$(1, 0)$

Imaginary Killing Spinor の case

(M, g) φ $\quad \nabla_X \varphi = \sqrt{-1}\frac{1}{2} X \cdot \varphi \quad (\forall X)$

$|\varphi|^2 = \langle \varphi, \varphi \rangle$ non-const fn (zero点なし)

$V^\varphi = \sqrt{-1} \sum \langle e_i \varphi, \varphi \rangle e_i$

conf vector field

$$L_{V^\varphi} g = 2 |\varphi|^2 g$$

さらに $V^\varphi \in \Omega^1(M) \cong \mathfrak{X}(M)$ テンソル解析

とみると $dV^\varphi = 0$. Tashiro 1965

Fact V : conformal vector field

$L_V g = 2fg$ $\quad (\cancel{h}\ f \in C^\infty(M))$ $\quad (\mathrm{div} V = nf)$

さらに $dV = 0$

N : V の zero 点の数

$\Rightarrow N \leqq 2$

(1) $N = 2$ $\quad (M, g) \simeq S^n$ (conf)

(2) $N = 1$ $\quad (M, g) \simeq \mathbb{R}^n$ (conf) $\quad f = h$

(3) $N = 0$ $\quad \tilde{M} = {}^\exists F \times \mathbb{R}$, $\quad g = h(t)^2 g_F + dt^2$

${}^\exists h : \mathbb{R} \to \mathbb{R}^*$, $\quad V = h(t) \frac{\partial}{\partial t}$

(3) の case, もう少し計算すると.

(M, g): imaginarly killing spinor をもつ.

$\Rightarrow (M, g) = (F^{n-1} \times \mathbb{R}, e^{-2t} g_F + dt^2)$

parallel spinor をもつ

1989

(by Baum)

$\lambda = \dfrac{f'}{2f}$

Note 質問があったので)

$$Cl(V) = T^*(V)/J(S') \xleftarrow{\pi} T^*(V)$$

$\pi|_V$, $\pi|_{\mathbb{R}}$ は inj

proof ちゃんとした proof を与える

$f : V \ni v \longmapsto v\wedge - \iota(v) \in \mathrm{End}(\Lambda^* V)$

ι interior product

$\rightsquigarrow$ $F : T^*(V) \to \mathrm{End}(\Lambda^* V)$ 1 extend

$f(v) f(v) = -\langle v, v\rangle$ をみたす

$$\therefore \quad \begin{tikzcd} T^*(V) \arrow[r, "F"] \arrow[d, "\pi"] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{End}(\Lambda^* V) \\ T^*(V)/J(S') \arrow[ur, "\tilde{F}"'] & \end{tikzcd}$$

($\tilde{F}$ の construction に, $\pi|_V$, $\pi|_{\mathbb{R}}$ inj は不要

であり, $T^*(V)/J(S')$ の正体がわからなく

ても o.k.)

$1 = F(1) = \tilde{F} \circ \pi(1)$ より $\pi(1) \neq 0$

$\therefore \quad \pi|_{\mathbb{R}}$ は inj

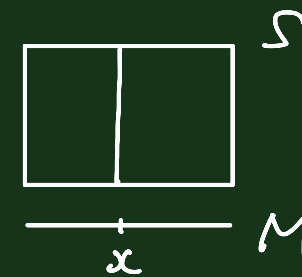
また, $F(v)\,1 = (v\wedge - \iota(v))\,1 = v$ $\quad \pi(v) = 0$ なら $v = 0$

$\therefore \ \pi|_V$ inj //

Def of R-S operator, R-S fields

(M, g) : Riem spin mfd

$S \to M$: Spinor bundle on M



$S \otimes TM \cong S \otimes T^*M$ $\quad (TM = S_1)$

$\cong S \oplus S_{3/2}$ $\quad$ irr decomp

D : Dirac op

$$\begin{tikzcd}
 & & \Gamma(S) \\
\Gamma(S) \arrow[rru, bend left, "D"] \arrow[rrd, bend right, "P"'] \arrow[r, "\nabla"] & \Gamma(S \otimes T^*M) \arrow[ru, "pr"] \arrow[rd, "pr"'] & \\
 & & \Gamma(S_{3/2})
\end{tikzcd}$$

$pr(\varphi \otimes e) = \frac{1}{\sqrt{n}} e \cdot \varphi$

P : Penrose (twistor) op

∇ "=" $D + P$

$$S_{3/2} \otimes T^*M \cong S_{3/2} \oplus S \oplus S_{5/2} \oplus S_{3/2,3/2}.$$

$\nabla = Q + P^* + P_{5/2} + P_{3/2,3/2}$

Q : Rarita-Schwinger op

$$\begin{tikzcd}
 & & \Gamma(S_{3/2}) \\
\Gamma(S_{3/2}) \arrow[r, "\nabla"] \arrow[urr, bend left, "Q"] \arrow[drr, bend right, "P^*"'] & \Gamma(S_{3/2} \otimes T^*M) \arrow[ur] \arrow[dr] & \\
 & & \Gamma(S)
\end{tikzcd}$$

P

P^* : adj of P

$$a Q^2 + b P P^* + c P^*_{5/2} P_{5/2} + d P^*_{3/2,3/2} P_{3/2,3/2} = \mathrm{Cur}$$

Q is 1-st order self ad elliptic, but $\sigma(Q^2) \neq \sigma(\Delta)$

P is overdetermined elliptic, i.e. P^*P is elliptic

$\hookrightarrow \Gamma(S_{3/2}) \cong \ker P^* \oplus \mathrm{Im}(P)$

R-S op via twisted Dirac op ⑨

$$D_{TM} : \Gamma(\underline{S \otimes TM}) \to \Gamma(S \otimes TM) \quad \text{by } \sum_{i=1}^{n} (e_i \cdot \otimes I) \nabla_{e_i}$$

$$S \otimes TM = S \oplus S_{3/2}$$

$$\therefore \quad D_{TM} = \begin{pmatrix} \frac{2-n}{n} D & 2P^* \\ 2/n\, P & Q \end{pmatrix} \quad ☆$$

twisted Lichnerowicz

$$D_{TM}^2 = \Delta + Scal/8 - I \otimes Ric = \begin{pmatrix} \Delta + \frac{n-8}{8n} Scal & 2\left(Ric - \frac{1}{n} g\right)^* \\ 2\left(Ric - \frac{1}{n} g\right) & \Delta + Scal/8 - Ric_{3/2} \end{pmatrix}$$

$$☆^2 \quad \| \qquad\qquad \|$$

$$\begin{pmatrix} \frac{(n-2)^2}{n^2} D^2 + \frac{4}{n} P^* P & 2\left(P^* Q - \frac{n-2}{n} D P^*\right) \\ \frac{2}{n}\left(QP - \frac{n-2}{n} PD\right) & Q^2 + \frac{4}{n^2} PP^* \end{pmatrix}$$

$$D_{TM} D_{TM}^2 = D_{TM}^2 D_{TM}$$

Prop (M, g) cpt Einstein spin mfd

$$\begin{cases} Q^2 + \frac{4}{n} PP^* = \Delta + \frac{n-8}{8n} \mathrm{Scal} \\ QP = \frac{n-2}{n} PD, \quad P^* Q = \frac{n-2}{n} DP^* \\ \Delta P = P\Delta, \quad P^*\Delta = \Delta P^*, \quad Q\Delta = \Delta Q \end{cases}$$

Δ standard Laplacian $= \nabla^* \nabla + \mathfrak{F}(R)$

$$\Gamma(S_{\frac{3}{2}}) = \ker P^* \oplus \mathrm{Im}\, P \circlearrowleft Q, \Delta$$

In general, $\Delta = \Delta_{3/2} \not\geq 0$ (similar to $\Delta_2 \not\geq 0$)

$\Downarrow$ vanishing thm is not easy.

$$Q^2 = \begin{cases} \Delta + \frac{n-8}{8n} \mathrm{scal} & \text{on } \ker P^* \\ \left(\frac{n-2}{n}\right)^2 (\Delta + \mathrm{Scal}/8) & \text{on } \mathrm{Im}\, P \end{cases}$$

where Δ: standard Lap (on G/K, Casimir)

Rarita-Schwinger field

φ : R-S field $\iff$ $\varphi \in \Gamma(S_{3/2})$, $D_{TM}\varphi = 0$

$\iff$ $\varphi \in \Gamma(S_{3/2})$, $Q\varphi = 0$, $P^*\varphi = 0$

$$RS(M) := \dim\{\varphi \mid \varphi : \text{R-S fields}\}$$

- $\dim M =$ even. cpt Einstein spin

$$RS(M) \geqq |RS^+(M) - RS^-(M)| = \left| \int_M \hat{A}(M)\, ch(TM) \right|$$

- M cpt Einstein spin $Scal \geqq 0$

$$\Rightarrow RS(M) = \dim \ker Q$$

RS-fields on cpt sym sp

$\Delta_{3/2}$ = Casimir op on G/K $\therefore$ $\Delta \geqq 0$ on G/K

vanishing thm G/K cpt type irr sym sp dim > 8
(∵ Einstein)

$\Rightarrow$ $RS(G/K) = 0$

∵) $$Q^2 = \underbrace{\Delta_{3/2}}_{\geqq 0} + \underbrace{\frac{n-8}{8n} Scal}_{>0}$$

Thm [H.S, 2019]

G/K irr cpt type symm space with $RS > 0$

$\Rightarrow$ $G/K = Gr_2(\mathbb{C}^4)$, $\mathbb{H}P^2$, $G_2/SO(4)$ ($\rightsquigarrow$ q-kähler)
$SU(3)$ ($PSU(3)$-str)

(M, g) cpt irr Riem spin

with $\nabla \varphi = 0$ $\quad \varphi \in \Gamma(S_{3/2})$

$\Rightarrow$ M sym sp. $\quad \mathrm{Hol}(M)$ in Berger's list

$$\left.\begin{array}{l} SO(n),\ U(m) \\ G_2,\ Spin(7) \\ SU(m) \\ Sp(1)Sp(k) \end{array}\right\} \quad S_{3/2} \text{ trivial comp をもたない}$$

$Sp(n)$ は $S_{3/2} = \bigoplus$

$n-1$ 個の trivial comp

Thm (Homma–Semmelmann)

(M^4, g) cpt irr Spin with

parallel R-S fields

$\Rightarrow (M, g) \cong$

$Gr_2(\mathbb{C}^4)$, $\mathbb{H}P^2$, $G_2/SO(4)$, $SU(3)$

or h-K mfd "