Four-dimensional Wess-Zumino-Witten actions

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Abstract

We shall give an axiomatic construction of Wess-Zumino-Witten actions valued in $G = SU(N)$, $N \geq 3$. It is realized as a functor WZ from the category of conformally flat four-dimensional manifolds to the category of line bundles with connection that satisfies, besides the axioms of a topological field theory, the axioms which abstract the characteristics of Wess-Zumino-Witten actions. To each conformally flat four-dimensional manifold Σ with boundary $\Gamma = \partial\Sigma$, a line bundle $L = WZ(\Gamma)$ with connection over the space ΓG of mappings from Γ to G is associated. The Wess-Zumino-Witten action is a non-vanishing horizontal section $WZ(\Sigma)$ of the pullback bundle r^*L over ΣG by the boundary restriction r. $WZ(\Sigma)$ is required to satisfy a generalized Polyakov-Wiegmann formula with respect to the pointwise multiplication of the fields ΣG. Associated to the WZW-action there is a geometric description of the extension of the Lie group $\Omega^3 G$ due to J. Mickelsson. In fact we have two abelian extensions of $\Omega^3 G$ that are in duality.

Keywords Wess-Zumino-Witten actions, Axiomatic field theories.

0   Introduction

In this paper we shall give an axiomatic construction of the Wess-Zumino-Witten action. Axiomatic approaches to field theories were introduced by G. Segal in two-dimensional conformal field theory (CFT), and by M. F. Atiyah in topological field theory, [1, 16]. The axioms abstract the functorial structure that the path integral would create if it existed as a mathematical object. Thus a CFT is defined as a Hilbert space representation of the operation of disjoint union and contraction on a category of manifolds with

parametrized boundaries. The functional integral formalism was also explored by Gawedzki [7] to explain the WZW conformal field theory. M. A. Singer [18] proposed a four-dimensional CFT in the language of Penrose's twistor space, where Riemann surfaces of two-dimensional CFT were replaced by conformally flat four-dimensional manifolds.

In a four-dimensional Wess- Zumino-Witten model the space of field configurations is the space of all maps from closed four-dimensional manifolds with or without boundary into a compact Lie group. We know from the discussions in [18, 21] that the geometric setting for CFT is most naturally given by the category of conformally flat manifolds. So we adopt this category of manifolds also for our WZW model. Let Σ be a conformally flat four-dimemsional manifold with boundary $\Gamma = \partial\Sigma$ which may be the empty set. Let $G = SU(N)$ with $N \geq 3$. The amplitude of the WZW model is given formally by the functional integration over fields $f \in \Sigma G = Map(\Sigma, G)$ with the boundary restriction equal to the prescribed $g \in \Gamma G = Map(\Gamma, G)$:

$$A_\Sigma(g) = \int_{f \in \Sigma G;\, f|\Gamma = g} \exp\{2\pi i S_\Sigma(f)\}\, \mathcal{D}f, \qquad (0.1)$$

where $S_\Sigma(f)$ is defined by;

$$S_\Sigma(f) = -\frac{ik}{12\pi^2} \int_\Sigma tr(df^{-1} \wedge *df) + C_\Sigma(f).$$

Since we deal with contributions that are topological in nature we omit the first term (kinetic term). The exponential of the second term

$$WZ(\Sigma)(f) = \exp\{2\pi i C_\Sigma(f)\} \qquad (0.3)$$

is called the Wess-Zumino-Witten action. (In [7, 8] it is called an amplitude or a probability amplitude. In [3] it is called the Wess-Zumino-Witten action.) When Σ has no boundary $C_\Sigma(f)$ is defined by

$$C_\Sigma(f) = \frac{i}{240\pi^3} \int_{B^5} tr(d\tilde{f} \cdot \tilde{f}^{-1})^5, \qquad (0.4)$$

where $\tilde{f}$ is an extension of f to a 5-dimensional manifold B^5 with boundary $\partial B^5 = \Sigma$. Since Σ is a compact conformally flat manifold it is the boundary of a five-dimensional manifold B^5. But it is not clear that we can take such a smooth extension of f over B^5. If Σ is simply connected it is conformally equivalent to a four-dimensional sphere, and then, since $\pi_4(G) = 1$, there exists a smooth extension of f to the five-dimensional disc D^5 and $C_{S^4}(f)$ is defined up to $\mathbf{Z}$, that is, $\exp\{2\pi i C_{S^4}(f)\}$ is well defined. The problem arises

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as to how to define the action $WZ(\Sigma)(f)$ for general Σ without boundary. On the other hand in (0.1) we are dealing with a four-manifold with boundary, so we must also give the definition of the action $WZ(\Sigma)(f)$ for Σ with non-empty boundary. The above discussions lead to the following conclusion: *A four-dimensional Wess-Zumino-Witten (WZW) model means to assign a proper definition of the action $WZ(\Sigma)(f)$ to every compact conformally flat four-manifold Σ with or without boundary.*

We shall construct the actions $WZ(\Sigma)$ as the objects that satisfy several axioms. Our WZW actions are associated to four-dimensional manifolds with boundary and respect the functorial properties of various operations on the basic manifolds. Hence we impose on $WZ(\Sigma)$ several axioms that are similar to those of topological field theories. Axioms of topological field theories were introduced by M. F. Atiyah in [1]. They apply to a functor from the category of topological spaces to the category of vector spaces. K. Gawedzki explored in the same spirit the axioms which characterize the amplitudes of two-dimensional WZW theory, [7]. Since our objects are not the amplitudes but the actions of the field, we describe our four-dimensional WZW theory as a functor WZ from the category of four-manifolds with boundary to the category of *complex line bundles*. This functor is required to satisfy the involutory axiom, the multiplicativity axiom and the associativity axiom that represent respectively the orientation reversal and the operations of disjoint union and contraction of the basic manifolds. Next we shall introduce two axioms that are characteristic of WZW models. We know that the action functional in field theory has topological effects, that is, it gives rise to the holonomy of a connection. So we require as our next axiom that the action $WZ(\Sigma)$ gives rise to a four-dimensional analogue of parallel transport associated to a connection of the complex line bundle. Higher-dimensional parallel transports as well as holonomies were discussed by Y. Terashima, [19], following the idea of Gawedzki in [8] that relates isomorphism classes of line bundles with connection and the $U(1)$-holonomy coming from WZW action. The fundamental property of the WZW action is its behavior under the pointwise multiplication of fields. It is expressed by the Polyakov-Wiegmann formula, [13], and its generalization to four-dimensional sphere was given by J. Mickelsson,[11]. As our last axiom we demand that $WZ(\Sigma)$ satisfies the generalized Polyakov-Wiegmann foumula over ΣG. More precisely the WZW actions can be stated as follows. A four-dimensional WZW model means a functor WZ that assigns to each manifold Σ, and its boundary $\Gamma = \partial\Sigma$, a line bundle $L = WZ(\Gamma)$ over the space of maps ΓG, and a non-vanishing section $WZ(\Sigma)$ over ΣG of the pullback line bundle r^*L by the boundary restriction map $r : \Sigma G \longrightarrow \Gamma G$. The functor WZ satisfies the axioms of topological field theories. We demand that each line bundle $WZ(\Gamma)$ has a connection

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and that $WZ(\Sigma)$ is parallel with respect to the induced connection on r^*L. We impose moreover that on r^*L there is defined a product which is equivariant with respect to the product on ΣG through the Polyakov-Wiegmann formula:

$$WZ(\Sigma)(fg) = WZ(\Sigma)(f) * WZ(\Sigma)(g) \qquad \text{for } f, g \in \Sigma G. \qquad (0.5)$$

We shall see that $WZ(\Sigma)$ is a positive integer for a compact Σ.

Here is a brief summary of each section. In section 1, we explain following [18] that the category of conformally flat manifolds fits most naturally the construction of axiomatic CFT and our WZW model. In 1.2 we introduce the axioms of our WZW model. Gawedzki in [7] gave two line bundles in duality over the loop space LG that correspond to the 2-cocycles obtained by transgressing the 3-curvature on G. In the same spirit we shall give in section 2 two line bundles $WZ(S^3)$ and $WZ((S^3)')$ in duality over $\Omega_0^3 G$ that correspond to the 2-cocycles obtained by transgressing the 5-form over G. Here $\Omega_0^3 G$ is the space of smooth maps from S^3 to G that have degree 0. In fact we have a two-form on $\Omega_0^3 G$;

$$\beta = \frac{i}{240\pi^3} \int_{S^3} tr(df \cdot f^{-1})^5, \qquad (0.6)$$

which generates the integral cohomology class $H^2(\Omega_0^3 G, \mathbf{Z})$. Hence it defines a line bundle with connection on $\Omega_0^3 G$, with the curvature β. This is $WZ(S^3)$. Let DG be the space of maps from a hemisphere D to G and let $D'G$ be the space of maps for the other hemisphere. We shall give a non-vanishing section $WZ(D)$ of the pullback line bundle of $WZ(S^3)$ by the boundary restriction map $r : DG \longrightarrow \Omega_0^3 G$. Intuitivvely $WZ(D)(f)$ is the holonomy associated to the curvature β over the four-dimensional path $f \in DG$. Similarly we have a non-vanishing section $WZ(D')$ of the pullback line bundle of $WZ((S^3)')$ by $r' : D'G \longrightarrow \Omega_0^3 G$. The connections on $WZ(S^3)$ and $WZ((S^3)')$ are given in 2.8, with respect to which $WZ(D)$ and $WZ(D')$ are parallel respectively. In section 3 we construct the functor WZ. The line bundle $WZ(\Gamma)$ is defined as the tensor product of $WZ(\Gamma_i)$ for each boundary component Γ_i parametrized by S^3, while each $WZ(\Gamma_i)$ is defined as the pullback of $WZ(S^3)$ or $WZ((S^3)')$ by the map $\Gamma_i G \longrightarrow S^3 G$ coming from the parametrization. The non-vanishing section $WZ(\Sigma)$ of $r^*WZ(\Gamma)$ is defined from the non-vanishing sections $WZ(D)$ and $WZ(D')$ by cutting and pasting methods and by using the dual relations, i.e. the associativity axiom. The connection on $WZ(\Gamma)$ is induced from those on $WZ(S^3)$ and $WZ((S^3)')$ by a standard procedure. WZ satisfies the axioms that abstract the functorial structure of the WZW actions. In particular we have

the Polyakov-Wiegmann formula generalized to ΣG for any conformally flat four-manifold Σ. In section 4 we shall discuss extensions of the Lie group $\Omega_0^3 G$. It is a well known observation that the two-dimensional WZW action gives a geometric description of central extensions of the loop group, [2, 7]. The $U(1)-$principal bundle over $\Omega_0^3 G$ associated to the line bundle $WZ(S^3)$ however does not have any group structure. Instead J. Mickelsson in [11] gave an extension of $\Omega_0^3 G$ by the abelian group $Map(\mathcal{A}_3, U(1))$, where $\mathcal{A}_3$ is the space of connections on S^3. We shall explain two extensions of Mickelsson's type that are dual to each other.

1 Axioms for a 4-dimensional WZW model

1.1

The basic components of four-dimensional CFT are some well behaved class of four-dimensional manifolds M with parametrized boundaries, together with the natural operations of disjoint union

$$(M_1, M_2) \longrightarrow M_1 \cup M_2,$$

and contraction

$$M \longrightarrow \tilde{M},$$

where $\tilde{M}$ is obtained from M using the parametrization to attach a pair of boundary three-spheres to each other. A four-dimensional CFT is then defined as a Hilbert space representation of the operation of disjoint union and contraction on these basic components. Now we know that the geometric setting for this CFT is most naturally given by the conformal equivalence classes of conformally flat four-dimensional manifolds. This fact was explained by M. A. Singer [18] , R. Zucchini [21] and Mickelsson-Scott [12].

Here we shall see following [18] the fact that the class of compact conformally flat four-dimensional manifolds with boundary is closed under the operation of sewing manifolds together across a boundary component. For any conformally flat M the developing map $M \longrightarrow S^4$ is a well defined conformal local diffeomorphism. A closed 3-manifold $N \subset M$ is called a *round* S^3 in M if it goes over diffeomorphically to a round S^3 in S^4 under development. This is well defined because the developing map is unique up to composition with conformal transformations. For standard M, the boundary ∂M consists of a disjoint union of round S^3s, [15]. For each boundary component B one can find a neighborhood of B in M and a conformal diffeomorphism of this neighborhood onto a neighborhood of the equator in the

northern hemisphere of S^4. If we have two boundary components B and $\tilde{B}$ of M and an orientation reversing conformal diffeomorphism $\psi : B \longrightarrow \tilde{B}$, then B and $\tilde{B}$ can be attached using ψ and the resulting manifold will have a unique conformally flat structure compatible with the original one on M.

1.2

Now we give the precise definition of a four-dimensional WZW model.

Let $\mathcal{M}_4$ be the conformal equivalence classes of all compact conformally flat four-dimensional manifolds M with boundary $\partial M = \bigcup_{i \in I} \Gamma_i$ such that each oriented component Γ_i is a round S^3, and is endowed with a parametrization $p_i : S^3 \longrightarrow \Gamma_i$. We distinguish positive and negative parametrizations $p_i : S^3 \longrightarrow \Gamma_i$, $i \in I_\pm$, depending on whether p_i respects the orientation of Γ_i or not.

Let $\mathcal{M}$ be the category whose objects are three-dimensional manifolds Γ which are disjoint unions of round S^3s. A morphism between three-dimensional manifolds Γ_1 and Γ_2 is an oriented cobordism given by $\Sigma \in \mathcal{M}_4$ with boundary $\partial \Sigma = \Gamma_2 \bigcup (\Gamma_1')$, where the upper prime indicates the opposite orientation.

Let $\mathcal{L}$ be the category of complex line bundles.

Let $G = SU(N), N \geq 3$. In the following the set of smooth mappings from a manifold M to G that are based at some point $p_0 \in M$ is denoted by $MG = Map(M, G)$. MG becomes a group under product of mappings. For a $\Sigma \in \mathcal{M}_4$ with boundary $\Gamma = \partial \Sigma$, r denotes the restriction map

$$r : \Sigma G \longrightarrow \Gamma G, \qquad r(f) = f|\Gamma. \tag{1.1}$$

A four-dimensional WZW model means a functor WZ from the category $\mathcal{M}$ to the category $\mathcal{L}$ which assigns:

WZ1,   to each manifold $\Gamma \in \mathcal{M}$, a complex line bundle $WZ(\Gamma)$ over the space ΓG,

WZ2,   to each $\Sigma \in \mathcal{M}_4$, with $\partial \Sigma = \Gamma$, a non-vanishing section $WZ(\Sigma)$ of the pullback line bundle $r^* WZ(\Gamma)$.

Recall that the pullback bundle is by definition

$$r^* WZ(\Gamma) = \{(f, u) \in \Sigma G \times WZ(\Gamma); \quad \pi u = r(f), \} \tag{1.2}$$

and the section $WZ(\Sigma)$ is given at $f \in \Sigma G$ by

$$WZ(\Sigma)(f) = (f, u) \quad \text{with } u \in \pi^{-1}(r(f)) = WZ(\Gamma)_{r(f)}.$$

WZ being a functor from $\mathcal{M}$ to $\mathcal{L}$, a conformal diffeomorphism $\alpha : \Gamma_1 \longrightarrow \Gamma_2$ induces an isomorphism $WZ(\alpha) : WZ(\Gamma_1) \longrightarrow WZ(\Gamma_2)$ such that $WZ(\beta\alpha) = WZ(\beta)WZ(\alpha)$ for $\beta : \Gamma_2 \longrightarrow \Gamma_3$. Also if α extends to a conformal diffeomorphism $\Sigma_1 \longrightarrow \Sigma_2$, with $\partial\Sigma_i = \Gamma_i$, $i = 1, 2$, then $WZ(\alpha)$ takes $WZ(\Sigma_1)$ to $WZ(\Sigma_1)$.

The functor WZ satisfies the following axioms. **A1**,**A2** and **A3** represent in the category of line bundles the orientation reversal and the operation of disjoint union and contraction. These axioms are stated in the same manner as in topological field theories, [1]. Axioms **A4** and **A5** are characteristic of the WZW model.

A1   (Involution):

$$WZ(\Gamma') = WZ(\Gamma)^* \tag{1.3}$$

where $*$ indicates the dual line bundle.

A2   (Multiplicativity):

$$WZ(\Gamma_1 \cup \Gamma_2) = WZ(\Gamma_1) \otimes WZ(\Gamma_2). \tag{1.4}$$

A3   (Associativity):

For a composite cobordism $\Sigma = \Sigma_1 \cup_{\Gamma_3} \Sigma_2$ such that $\partial\Sigma_1 = \Gamma_1 \cup \Gamma_3$ and $\partial\Sigma_2 = \Gamma_2 \cup \Gamma_3'$, we have

$$WZ(\Sigma)(f) = < WZ(\Sigma_1)(f_1), WZ(\Sigma_2)(f_2) >, \tag{1.5}$$

for any $f \in \Sigma G$, $f_i = f|\Sigma_i$, i=1,2, where $< , >$ denotes the natural pairing

$$WZ(\Gamma_1) \otimes WZ(\Gamma_3) \otimes WZ(\Gamma_3') \otimes WZ(\Gamma_2) \longrightarrow WZ(\Gamma_1) \otimes WZ(\Gamma_2). \tag{1.6}$$

More precisely, let $WZ(\Sigma_1)(f_1) = (f_1, u_1 \otimes v)$ and $WZ(\Sigma_2)(f_2) = (f_2, u_2 \otimes v')$ with $u_i \in WZ(\Gamma_i)$ for $i = 1, 2$, and $v \in WZ(\Gamma_3))$, $v' \in WZ(\Gamma_3')$. From the definition $u_i \in \pi^{-1}(f_i|\Gamma_i)$, $v \in \pi^{-1}(f_1|\Gamma_3)$ and $v' \in \pi^{-1}(f_2|\Gamma_3')$. On the other hand, let $WZ(\Sigma)(f) = (f, w_1 \otimes w_2) \in WZ(\Gamma_1) \otimes WZ(\Gamma_2)$ with $w_i \in \pi^{-1}(f|\Gamma_i)$, for $i = 1, 2$. Then axiom **A3** says that $w_1 \otimes w_2 = < v', v > u_1 \otimes u_2$. The multiplicative axiom **A2** asserts that if $\partial\Sigma = \Gamma_2 \bigcup (\Gamma_1')$, then $WZ(\Sigma)$ is a section of

$$r_1^* WZ(\Gamma_1') \otimes r_2^* WZ(\Gamma_2) = Hom(r_1^* WZ(\Gamma_1), r_2^* WZ(\Gamma_2)). \tag{1.7}$$

Therefore any cobordism Σ between Γ_1 and Γ_2 induces a homomorphism of sections of pullback line bundles

$$WZ(\Sigma) : C^\infty(\Sigma, r_1^* WZ(\Gamma_1)) \longrightarrow C^\infty(\Sigma, r_2^* WZ(\Gamma_2)). \tag{1.8}$$

We impose:

1.
$$WZ(\phi) = \mathrm{C} \quad \text{for } \phi \text{ the empty 3-dimensional manifold,} \qquad (1.9)$$

2.
$$WZ(S^4) = 1 \qquad (1.10)$$

3.
$$WZ(\Gamma \times [0,1]) = Id.(WZ(\Gamma) \longrightarrow WZ(\Gamma)). \qquad (1.11)$$

Corollary 1.1. *If Σ has no boundary ($\partial\Sigma = \phi$), then $WZ(\Sigma) \in \mathrm{C}$.*

The following axioms are characteristic of WZW models.

A4 For each $\Sigma \in \mathcal{M}_4$ with $\Gamma = \partial\Sigma$, $WZ(\Gamma)$ has a connection, and $WZ(\Sigma)$ is parallel with respect to the induced connection on $r^*WZ(\Gamma)$.

A5 (Generalized Polyakov-Wiegmann formula): For each $\Sigma \in \mathcal{M}_4$ with $\Gamma = \partial\Sigma$, on the pullback line bundle $r^*WZ(\Gamma)$ is defined a product $*$ with respect to which we have

$$WZ(\Sigma)(fg) = WZ(\Sigma)(f) * WZ(\Sigma)(g) \qquad \text{for any } f, g \in \Sigma G. \quad (1.12)$$

The well known Polyakov-Wiegmann formula extended by J. Mickelsson [11] is concerned with the case of the four-dimensional sphere, $\Sigma = S^4$.

From now on we shall construct the functor WZ step by step. In section 2.5 we shall construct two line bundles over S^3G, which are $WZ(S^3)$ and $WZ((S^3)')$. In section 3 we give the functor WZ of WZW actions step by step starting from $WZ(S^3)$ and $WZ((S^3)')$.

2 Line bundles on Ω^3G

2.1

In the following we denote by Ω^3G, instead of S^3G, the set of smooth mappings f from a S^3 to $G = SU(N)$ that are based, i.e., $f(p_o) = 1$, at some point $p_o \in S^3$. It is known that Ω^3G is not connected and is divided into denumerable sectors labelled by the soliton number (the mapping degree). Here we follow the explanation due to I. M. Singer of these facts [17], see also [4, 10]. Let the evaluation map, $ev : S^3 \times \Omega^3G \longrightarrow G$, be defined by $ev(m, \varphi) = \varphi(m), m \in S^3, \varphi \in \Omega^3G$. The Maurer-Cartan form $g^{-1}dg$ on G gives the identification of the tangent space T_eG at $e \in G$ and

$Lie\,G = su(N)$. The primitive generators of the cohomology $H^*(G, \mathbb{R})$ are given by

$$\omega_3 = -\frac{1}{4\pi^2} tr(g^{-1}dg)^3, \quad \omega_5 = \frac{-i}{2\pi^2} tr(g^{-1}dg)^5, \quad \cdots . \tag{2.1}$$

Integration on S^3 of the pull back of ω_{2k-1} by the evaluation map ev gives us the following $2(k-2)$ form on $\Omega^3 G$;

$$\nu_{2k-1} = (\frac{1}{2\pi i})^k \frac{((k-1)!)^2}{(2k-1)!} \int_{S^3} tr(d\varphi\,\varphi^{-1})^{2k-1}, \quad 3 \le 2k-1 \le 2N-1. \tag{2.2}$$

In particular ν_3 is the mapping degree of φ;

$$\deg \varphi = \frac{i}{24\pi^2} \int_{S^3} tr(d\varphi\,\varphi^{-1})^3. \tag{2.3}$$

Proposition 2.1.   *1.*

$$S^3 Lie\,G \xrightarrow{\exp} \Omega^3 G \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

is exact.

2.
$$\deg \varphi_1 \cdot \varphi_2 = \deg \varphi_1 + \deg \varphi_2.$$

See [4, 10].

2.2

Let P_G be a G-principal bundle over S^4. Let $\mathcal{A}$ be the space of connections on P_G, that are $Lie\,G$-valued one-forms on P_G. Let $\mathcal{G} = S^4 G$ be the group of based gauge transformations. The action of $\mathcal{G}$ on $\mathcal{A}$ is given by $A_g = g^{-1}Ag + g^{-1}dg$ for $A \in \mathcal{A}$ and $g \in \mathcal{G}$. $F = F(A) = dA + A^2$ denotes the curvature two-form of A.

The Chern-Simons form on P_G is

$$\omega_5^0(A) = tr\,(AF^2 - \frac{1}{2}A^3 F + \frac{1}{10}A^5). \tag{2.4}$$

We have then $tr(F^3) = d\omega_5^0(A)$.

From [22] we know the relation

$$\omega_5^0(A_g) - \omega_5^0(A) = d\alpha_4(A; g) + \frac{1}{10} tr(dg \cdot g^{-1})^5,$$

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with

$$\alpha_4(A; g) = tr[-\frac{1}{2}V(AF + FA - A^3) + \frac{1}{4}(VA)^2 + \frac{1}{2}V^3A], \qquad (2.6)$$

where $V = dg \cdot g^{-1}$.

Let D^5 be a five dimensional disc with boundary $\partial D^5 = S^4$. Integration over D^5 gives us the *gauge anomaly* :

$$\begin{aligned}
\Gamma(A, g) &= \frac{i}{48\pi^3} \int_{S^4} tr[-V(AF + FA - A^3) + \frac{1}{2}(VA)^2 + V^3A] + C_5(g), \\
C_5(g) &= \frac{i}{240\pi^3} \int_{D^5} tr(dg \cdot g^{-1})^5, \qquad (2.7)
\end{aligned}$$

here $g \in S^4G$ is extended to D^5G, in fact, we have such an extension by virtue of $\pi_4(G) = 1$. $C_5(g)$ may depend on the extension but it can be shown that the difference of two extensions is an integer, and $\exp(2\pi i C_5(g))$ is independent of the extension.

We put, for $f, g \in S^4G$,

$$\begin{aligned}
\gamma(f, g) &= \frac{i}{24\pi^3} \int_{S^4} \alpha_4(f^{-1}df, g) \\
&= \frac{i}{48\pi^3} \int_{S^4} tr[(dgg^{-1})(f^{-1}df)^3 + \frac{1}{2}(dgg^{-1}f^{-1}df)^2 + \\
&\qquad + (dgg^{-1})^3(f^{-1}df)]. \qquad (2.8)
\end{aligned}$$

and
$$\omega(f, g) = \Gamma(f^{-1}df, g) = \gamma(f, g) + C_5(g). \qquad (2.9)$$

Remark 2.1. Here we shall look at Mickelson's 2-cocycle for his abelian extension of Ω^3G, [11]. The cochain α_4 in (2-5) is a one-cochain on the group S^4G, valued in $Map(\mathcal{A}_4, \mathrm{R})$. The coboundary $\delta\alpha_4$ is given by

$$\begin{aligned}
\delta\alpha_4(A : g_1, g_2) &= d\beta + \alpha_4(g_1^{-1}dg_1; g_2) \\
\beta(A; g_1, g_2) &= -tr[\frac{1}{2}(dg_2g_2^{-1})(g_1^{-1}dg_1)(g_1^{-1}Ag_1) - \frac{1}{2}(dg_2g_2^{-1})(g_1^{-1}Ag_1)(g_1^{-1}dg_1)].
\end{aligned}$$

Mickelson's 2-cocycle $\gamma_\Delta(A; f, g)$ is defined as the integration of $\delta\alpha_4(A; g_1, g_2)$ over any region $\Delta \subset S^4$:

$$\gamma_\Delta(A; f, g) = \frac{i}{24\pi^3} \int_\Delta \delta\alpha_4(A; f, g). \qquad (2.10)$$

But for $\Delta = S^4$ it is independent of A and

$$\begin{aligned}
\gamma_{S^4}(A; f, g) &= \int_{S^4} \delta\alpha_4(A; f, g) \\
&= \int_{S^4} \alpha_4(f^{-1}df, g) = \gamma(f, g), \qquad (2.11)
\end{aligned}$$

for $f, g \in S^4 G$. Hence, instead of $\gamma_{S^4}(A; f, g)$, we use more simple $\gamma(f, g)$ for our purpose.

Remark 2.2. We have

$$\gamma(F, G) = \gamma_D(A; F, G) + \gamma_{D'}(A; F, G), \qquad (2.12)$$

for any $A \in \mathcal{A}_4$. Here D is an oriented hemisphere of S^4 and D' is the other hemisphere: $D \cup D' = S^4$.

Lemma 2.2 (Polyakov-Wiegmann). *For f, $g \in S^4 G$ we have*

$$C_5(fg) = C_5(f) + C_5(g) + \gamma(f, g) \qquad \mathrm{mod}\ \mathbf{Z}. \qquad (2.13)$$

The following formula was proved by Mickelsson in Lemma 4.3.7 of his book [10] .

$$C_5(fg) = C_5(f) + C_5(g) + \gamma_{S^4}(A; f, g) \qquad \mathrm{mod}\ \mathbf{Z}.$$

Since $\gamma_{S^4}(A; f, g) = \gamma(f, g)$ from (2.10) we have the proposition.

2.3

Now we are prepared to define the line bundle $WZ(\phi)$ over $Map(\partial S^4, G) = \phi$, and the section $WZ(S^4)$ of the pullback line bundle of $WZ(\phi)$ by the empty restriction map $r : S^4 G \longrightarrow \phi$.

Let L_ϕ be the quotient of $S^4 G \times \mathbf{C}$ by the equivalence relation;

$$(f, c) \sim (g, c\exp\{2\pi i\omega(f, f^{-1}g)\}). \qquad (2.15)$$

Then L_ϕ is a line bundle over $Map(\partial S^4, G) = \phi$ with the transition function $\exp\{2\pi i\omega(f, f^{-1}g)\}$, which we shall define as $WZ(\phi)$. Recall that $S^4 G$ is contractible. We have then

$$WZ(\phi) \simeq \mathbf{C}. \qquad (2.16)$$

The isomorphism is given by

$$[f, c] \longrightarrow c\exp\{-2\pi iC_5(f)\}.$$

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It is well defined because of the Polyakov-Wiegmann formula. Let $r^*WZ(\phi)$ be the pullback line bundle of $WZ(\phi)$ by the empty restriction map $r : S^4G \longrightarrow \phi$. The section $WZ(S^4)$ of $r^*WZ(\phi)$ over any $f \in S^4G$ is given by

$$WZ(S^4)(f) = [f, \exp\{2\pi i C_5(f)\}] \in WZ(\phi). \qquad (2.17)$$

By the isomorphism of (2.14) we can also write

$$WZ(S^4) = 1 \in \mathrm{C}.$$

We can define the product on the line bundle $WZ(\phi) \simeq \mathrm{C}$ in an obvious way, but we shall look this product more precisely, rather superfluously, for the sake of later sections. In $S^4G \times \mathrm{C}$ we define the product by putting;

$$(f, a) * (g, b) = (fg, ab \exp\{2\pi i \gamma(f, g)\}). \qquad (2.19)$$

Since the equivalence relation (2.13) respects the product, it gives a product on the line bundle $WZ(\phi)$. The Polyakov-Wiegmann formula (2.12) is stated as follows.

$$WZ(S^4)(fg) = WZ(S^4)(f) * WZ(S^4)(g) \qquad \text{for } f, g \in S^4G. \qquad (2.20)$$

2.4

In this paragraph we shall prepare some notations, definitions and elementary properties that will be used in the following sections.

Let $\Omega^3 G$ be as before the set of smooth mappings from S^3 to $G = SU(N)$ that are based. $\Omega^3 G$ is not connected but divided into the connected components by deg. We put

$$\Omega_0^3 G = \{g \in \Omega^3 G; \deg g = 0\}. \qquad (2.21)$$

The oriented 4-dimensional disc with boundary S^3 is denoted by D, while that with oposite orientation is denoted by D'. The composite cobordism of D and D' becomes S^4. We write as before $DG = Map(D, G)$ and $D'G = Map(D', G)$. The restriction to S^3 of a $f \in DG$ has degree 0; $f|S^3 \in \Omega_0^3 G$.

For an $a \in \Omega_0^3 G$ we denote by Da the set of those $g \in DG$ that is a smooth extension of a, respectively $D'a$ is the set of those $g' \in D'G$ that is a smooth extension of a. For $f \in Da$ and $g \in Db$ one has $fg \in D(ab)$, and every element of $D(ab)$ is of this form. Similarly for $D'(ab)$. We denote by $g \vee g' \in S^4G$ the map obtained by sewing $g \in DG$ and $g' \in D'(g|S^3)$.

The upper prime will indicate that the function expressed by the letter is defined on D', for example, $1'$ is the constant function $D' \ni x \longrightarrow 1'(x) = e \in G$, while 1 is the constant function $D \ni x \longrightarrow 1(x) = e \in G$. We write

$$D'f = \{f' \in D'G : f'|S^3 = f|S^3\}, \qquad Df' = \{f \in DG : f|S^3 = f'|S^3\}.$$

Let $f, g \in DG$ and $f|S^3 = g|S^3$. From (2.7) and (2.8) we see that $\gamma(f \vee f', f^{-1}g \vee 1')$ and $\omega(f \vee f', f^{-1}g \vee 1')$ are independent of $f' \in Df$;

$$\gamma(f \vee f', f^{-1}g \vee 1') = \frac{i}{48\pi^3} \int_D tr[(dgg^{-1})(f^{-1}df)^3 + \frac{1}{2}(dgg^{-1}f^{-1}df)^2 +$$
$$+ (dgg^{-1})^3(f^{-1}df)]. \tag{2.22}$$

Similarly, for $f', g' \in D'G$ such that $f'|S^3 = g'|S^3$. $\gamma(g \vee g', 1 \vee (g')^{-1}f')$ and $\omega(g \vee g', 1 \vee (g')^{-1}f')$ are independent of $g \in Dg'$. Hence $\exp\{2\pi i\omega(f \vee \cdot, f^{-1}g \vee 1')\}$ and $\exp\{2\pi i\omega(\cdot \vee f', 1 \vee (f')^{-1}g')\}$ are constants of $U(1)$.

Definition 2.1. 1. We put, for $f, g \in DG$ such that $f|S^3 = g|S^3$,

$$\chi(f, g) = \exp\{2\pi i\omega(f \vee \cdot, f^{-1}g \vee 1')\}. \tag{2.23}$$

2. We put, for $f', g' \in D'G$ such that $f'|S^3 = g'|S^3$,

$$\chi'(f', g') = \exp\{2\pi i\omega(\cdot \vee f', 1 \vee (f')^{-1}g')\}. \tag{2.24}$$

Lemma 2.3. *1. For $f, g \in DG$ such that $f|S^3 = g|S^3$, we have $\chi(f, g) \in U(1)$ and*

$$\exp\{2\pi iC_5(g \vee f')\} = \exp\{2\pi iC_5(f \vee f')\}\chi(f, g) \qquad \textit{for any } f' \in D'f \tag{2.25}$$

2. For $f', g' \in D'G$ such that $f'|S^3 = g'|S^3$, we have $\chi'(f', g') \in U(1)$ and

$$\exp\{2\pi iC_5(f \vee g')\} = \exp\{2\pi iC_5(f \vee f')\}\chi'(f', g'), \qquad \textit{for any } f \in Df' \tag{2.26}$$

The lemma follows from the Polyakov-Wiegmann formula.

2.5

Now we shall give two line bundles on $\Omega_0^3 G$ that are dual to each other. We shall follow the arguments due to K. Gawedzki [7], that were developed to construct two line bundles in duality over the loop group LG and to give the definition of WZW action on a hemisphere.

We consider the following quotient;

$$L = D'G \times \mathrm{C}/ \sim', \qquad (2.27)$$

where $\sim'$ is the equivalence relation defined by

$$(f', c') \sim' (g', d') \text{ if and only if } \begin{cases} f'|S^3 & = g'|S^3 \\ d' & = c'\chi'(f', g') \end{cases} \qquad (2.28)$$

The equivalence class of (f', c') is denoted by $[f', c']$. We define the projection

$$\pi : L \longrightarrow \Omega_0^3 G$$

by $\pi([f', c']) = f'|S^3$. L becomes a line bundle on $\Omega_0^3 G$ with the transition function $\chi'(f', g')$.

More precisely, let $a \in \Omega_0^3 G$ and take $f' \in D'a$. A coordinate neighborhood of a is given by

$$
\begin{aligned}
U_{f'} &= \{g'|S^3; \quad g' \in V_{f'}\} \\
V_{f'} &= \{g' \in D'G, \ g' = \exp X \cdot f'; \quad X \in D'(Lie\,G), \|X\| < \delta\,\}.
\end{aligned}
$$

The local trivialization of L is given by the map $\pi^{-1}(U_{f'}) \ni [h', c'] \longrightarrow (h'|S^3, c')$;

$$\pi^{-1}(U_{f'}) \simeq U_{f'} \times \mathrm{C}.$$

The transition function $\chi_{U_{f'},U_{g'}}(b)$ of L at $b \in U_{f'} \cap U_{g'}$ becomes as follows. Let $b \in U_{f'} \cap U_{g'}$. Let $h' \in V_{f'}$ and $k' \in V_{g'}$ be such that $h'|S^3 = k'|S^3 = b$. For $\xi = [h', c'] = [k', d'] \in \pi^{-1}(b)$ we have obviously $d' = \chi'(h', k')c'$. Hence

$$\chi_{U_{f'},U_{g'}}(b) = \chi'(h', k') \qquad (2.30)$$

The line bundle L is what we wanted to construct and will be denoted by $WZ(S^3)$.

In regard to the involution axiom **A1** which $WZ(\cdot)$ is required to satisfy we must define another line bundle on $\Omega_0^3 G$ corresponding to S^3 with opposite orientation. This line bundle $WZ((S^3)')$ is defined by

$$WZ((S^3)') = DG \times \mathrm{C}/ \sim \qquad (2.31)$$

with the equivalence relation

$$(f, c) \sim (g, d) \text{ if and only if } \begin{cases} f|S^3 & = g|S^3 \\ d & = c\chi(f, g) \end{cases} \qquad (2.32)$$

The projection $\pi : WZ((S^3)') \longrightarrow \Omega_0^3 G$ is given by $[f, c] \longrightarrow f|S^3$. It is a line bundle with the transition function $\chi(f, g)$.

$WZ(S^3)$ and $WZ((S^3)')$ are in duality so that the involution axiom **A1** is verified for these line bundles. In fact, the duality

$$WZ(S^3) \times WZ((S^3)') \longrightarrow C$$

is defined by

$$< [f', c'], [f, c] >= cc' \exp\{-2\pi i C_5(f \vee f')\}, \qquad (2.33)$$

where $f|S^3 = f'|S^3 \in \Omega_0^3 G$. If we note the evident fact that $\gamma(F, 1 \vee h')$ (resp. $\gamma(F, h \vee 1')$) in (2.19) is given by an integration over D' (resp. D), we see that the product of transition rules (2.25) and (2.28) imply the transition rule (2.13) of $WZ(\phi)$;

$$\chi(f, g)\chi'(f', g') = \exp\{2\pi i \omega(f \vee f', f^{-1}g \vee (f')^{-1}g')\}, \qquad (2.34)$$

Hence

$$WZ(S^3) \otimes WZ((S^3)') = WZ(\phi). \qquad (2.35)$$

Composed with (2-14) this implies the above duality.

2.6

Let $r : DG \longrightarrow S^3 G$ and $r' : D'G \longrightarrow S^3 G$ be the restriction maps.

We put, for $f \in DG$,

$$WZ(D)(f) = [f', \exp\{2\pi i C_5(f \vee f')\}] \in WZ(S^3)|_{r(f)}. \qquad (2.36)$$

Then we see from Lemma 2.3 that $WZ(D)$ gives a non-vanishing section of the pullback line bundle $r^*WZ(S^3)$.

In the same way we put, for $f' \in D'G$,

$$WZ(D')(f') = [f, \exp\{2\pi i C_5(f \vee f')\}] \in WZ((S^3)')|_{r'(f')}. \qquad (2.37)$$

$WZ(D')$ defines a non-vanishing section of $(r')^*WZ((S^3)')$.

Proposition 2.4. *For $f \in DG$ and $f' \in D'G$ such that $f|S^3 = f'|S^3$.*

$$< WZ(D)(f), WZ(D')(f') >= WZ(S^4)(f \vee f') \qquad (2.38)$$

In fact both sides are equal to $\exp\{2\pi i C_5(f \vee f')\}$.

2.7

The total space of the pullback bundle bundle $r^*WZ(S^3)$ is written as

$$r^*WZ(S^3) = \left\{ (f, \lambda);\ f \in DG,\ \lambda = [f', c'] \in WZ(S^3)_{r(f)} \right\}.$$

We define the product in $r^*WZ(S^3)$ by the formula;

$$(f, \lambda) * (g, \mu) = (fg, \nu), \tag{2.40}$$

where, for $\lambda = [f', a'] \in WZ(S^3)_{r(f)}$ and $\mu = [g', b'] \in WZ(S^3)_{r(g)}$, $\nu = [f'g', c'] \in WZ(S^3)_{r(fg)}$ is defined by

$$c' = a'b' \exp\{2\pi i \gamma(f \vee f', g \vee g')\}. \tag{2.41}$$

ν does not depend on the representations of λ and μ, and the product is well defined.

We have

$$WZ(D)(fg) = WZ(D)(f) * WZ(D)(g) \qquad \text{for } f, g \in DG. \tag{2.42}$$

In fact, this follows from the definition

$$WZ(D)(f) = [f', \exp\{2\pi i C_5(f \vee f')\}]$$

and the Polyakov-Wiegmann formula.

Similarly we have the product on $(r')^*WZ((S^3)')$ over $D'G$. It is given by

$$(f', \alpha) * (g', \beta) = (f'g', \gamma), \tag{2.43}$$

where, for $\alpha = [f, a] \in WZ((S^3)')_{r'(f')}$ and $\beta = [g, b] \in WZ((S^3)')_{r'(g')}$, $\gamma = [fg, c] \in WZ((S^3)')_{r'(f'g')}$ is defined by

$$c = ab \exp\{2\pi i \gamma(f \vee f', g \vee g')\}. \tag{2.44}$$

We have

$$WZ(D')(f'g') = WZ(D')(f') * WZ(D')(g') \qquad \text{for } f', g' \in D'G. \tag{2.45}$$

We note that product operations on $r^*WZ(S^3)$ and on $(r')^*WZ((S^3)')$ are compatible with the duality

$$r^*WZ(S^3) \times (r')^*WZ((S^3)') \longrightarrow WZ(\phi) \simeq \mathrm{C}, \tag{2.46}$$

that is, for (f, λ), $(g, \mu) \in r^*WZ(S^3)$ and for (f', λ'), $(g', \mu') \in (r')^*WZ((S^3)')$ such that $r(f) = r'(f')$ and $r(g) = r'(g')$, we have:

$$< (f, \lambda) * (g, \mu),\ (f', \lambda') * (g', \mu') > = < \lambda, \lambda' > * < \mu, \mu' >, \tag{2.47}$$

the right-hand side being the product in $WZ(\phi) \simeq \mathrm{C}$.

2.8

Next we define connections on $WZ(S^3)$. They are described as follows. Let $b \in \Omega_0^3 g$ and $U_{f'}$ be a coordinate neighborhood described in 2.5. On $U_{f'}$ we put

$$\theta_{U_{f'}}(b)(X) = \frac{i}{48\pi^3} \int_{D'} tr(h^{-1}dh)^3 \, dX, \qquad (2.48)$$

for $h \in D'b$ and $X \in D'(Lie\, G)$. We have

$$\theta_{U_{g'}} = \theta_{U_{f'}} + (\chi_{U_{f'},U_{g'}})^{-1} d\chi_{U_{f'},U_{g'}},$$

where $\chi_{U_{f'},U_{g'}}$ is the transition function of $WZ(S^3)$:

$$\chi_{U_{f'},U_{g'}}(b) = \chi'(h',k'),$$

for $h' \in D'b \cap V_{f'}$ and $k' \in D'b \cap V_{g'}$. We have a well defined connection θ on $WZ(S^3)$. The curvaure of θ becomes

$$F(X,Y) = -\frac{1}{24\pi^3} \int_{S^3} tr(V^2(XdY - YdX)), \qquad V = df\, f^{-1}|S^3. \quad (2.49)$$

The calculation for these formula is the same as in [6, 10, 11].

Similarly we have a connection on $WZ((S^3)')$ represented by a formula parallel to (2.43) but integrated on D.

On the pullback bundle $r^*WZ(S^3)$ there is an induced covariant derivative:

$$(r^*\bigtriangledown)_X s(f) = (\bigtriangledown_{r_*X} r_* s)(r(f)),$$

where r_*s is the section of $WZ(S^3)$ defined by $r_*s(b) = s(f) = [f',c'] \in WZ(S^3)_b$ for a (and any) $f \in Db$. X is a vector field on D, hence r_*X is a vector field on S^3.

Similarly the covariant derivative on $WZ((S^3)')$ is defined.

The sections $WZ(D)$ and $WZ(D')$ are parallel with respect to the respective covariant derivation. This follows almost from the definitions by virtue of the infinitesimal form of the Polyakov-Wiegman formula:

$$\frac{d}{dt}\Big|_{t=0} C_5(f\, e^{tX}) = \frac{i}{48\pi^3} \int_{S^4} tr(f^{-1}df)^3 \, dX, \text{ for } X \in S^4(Lie\, G), f \in S^4 G.$$

$$(2.50)$$

Proposition 2.5.

$$\bigtriangledown WZ(D) = 0 \qquad (2.51)$$
$$\bigtriangledown WZ(D') = 0 \qquad (2.52)$$

Remark 2.3. We could consider in the following construction of the WZW model those line bundles $WZ_n(S^3)$ associated to the n-th sector of $\Omega^3 G$, but for a fixed n. However in the sequel we shall restrict our discussion only to the contractible component $\Omega_0^3 G$.

3 Construction of WZW actions

3.1

Let $\Sigma \in \mathcal{M}_4$. Then Σ is a conformally flat manifold with boundary $\partial \Sigma = \Gamma = \bigcup_{i \in I_+} \Gamma_i \cup \bigcup_{i \in I_-} \Gamma_i$ with Γ_i a parametrized round S^3 in Σ.

For a $i \in I_- \oplus I_+$, the parametrization defines the map $p_i : S^3 \longrightarrow \Gamma_i$, and the map $p_i : \Gamma_i G \longrightarrow \Omega^3 G$, which we denote by the same letter. Then we have the pull-back bundle of $WZ(S^3)$ (resp. $WZ((S^3)')$) by p_i. We define

$$
\begin{aligned}
WZ(\Gamma_i) &= p_i^* WZ(S^3) \quad \text{for } i \in I_-, \\
WZ(\Gamma_i) &= p_i^* WZ((S^3)') \quad \text{for } i \in I_+,
\end{aligned} \tag{3.1}
$$

then we have respectively

$$
\begin{aligned}
WZ(\Gamma_i') &= p_i^* WZ((S^3)') \quad \text{for } i \in I_-, \\
WZ(\Gamma_i') &= p_i^* WZ(S^3) \quad \text{for } i \in I_+.
\end{aligned} \tag{3.2}
$$

The line bundle $WZ(\Gamma)$ is defined by

$$
WZ(\Gamma) = \otimes_{i \in I_-} WZ(\Gamma_i) \otimes \otimes_{i \in I_+} WZ(\Gamma_i). \tag{3.3}
$$

Now let $\alpha : S^3 \longrightarrow S^3$ be the restriction on S^3 of a conformal diffeomorphism on S^4. First we suppose that α preserves the orientation. Then, since the transition function χ is invariant under α, the line bundle $WZ(S^3)$ is invariant under α. If α reverses the orientation then D is mapped to D' and χ is changed to χ'. Then $\alpha^* WZ(S^3)$ becomes $WZ((S^3)')$. On the other hand the parametrizations p_i are uniquely defined up to composition with conformal diffeomorphisms. Therefore $WZ(\Gamma)$ is well defined for the conformal equivalence class of $\Gamma \in \mathcal{M}$.

The dual of $WZ(\Gamma)$ is

$$
WZ(\Gamma') = \otimes_{i \in I_-} WZ(\Gamma_i') \otimes \otimes_{i \in I_+} WZ(\Gamma_i'), \tag{3.4}
$$

and the duality; $WZ(\Gamma) \times WZ(\Gamma') \longrightarrow \mathrm{C}$, is given from (2.29) by:

$$
\begin{aligned}
&< \otimes_{i \in I_-} [f_i', c_i'] \otimes \otimes_{i \in I_+} [g_i, d_i] \,, \, \otimes_{i \in I_-} [f_i, c_i] \otimes \otimes_{i \in I_+} [g_i', d_i'] > \\
&= \Pi_{i \in I_-} c_i c_i' \cdot \Pi_{i \in I_+} d_i d_i' \cdot \exp\{-2\pi i \sum_{i \in I_-} C_5(f_i \vee f_i') - 2\pi i \sum_{i \in I_+} C_5(g_i \vee g_i')\}.
\end{aligned}
$$

The above defined $WZ(\Gamma)$ satisfies axioms **A1** and **A2**.

3.2

In the following we shall define step by step the section $WZ(\Sigma)$ of $r^*WZ(\Gamma)$ for any $\Sigma \in \mathcal{M}_4$ with the boundary $\partial\Sigma = \Gamma$ and $r : \Sigma G \longrightarrow \Gamma G$.

We obtain a compact manifold $\Sigma^c \in \mathcal{M}_4$ without boundary by sewing a copy D_i of D along Γ_i for $i \in I_-$ and a copy D'_i of D' for $i \in I_+$;

$$\Sigma^c = (\cup_{i\in I_-} D_i) \cup_{\cup_{I_-}\Gamma_i} \Sigma \cup_{\cup_{I_+}\Gamma_i} (\cup_{i\in I_+} D'_i).$$

For each boundary component Γ_i of Γ the parametrization p_i is extended to a parametrization $\tilde{p}_i : D_i \longrightarrow D$ if $i \in I_-$, and $\tilde{p}_i : D_i \longrightarrow D'$ if $i \in I_+$. The extension is unique up to composition with conformal transformations, see 1.1.

We put

$$WZ(D_i) = (\tilde{p}_i)^*WZ(D) \qquad (3.5)$$
$$WZ(D'_i) = (\tilde{p}_i)^*WZ(D'). \qquad (3.6)$$

For $i \in I_-$, $WZ(D_i)$ is a section of the pullback bundle of $WZ(\Gamma_i)$ by the restriction map $r_i : D_i G \longrightarrow \Gamma_i G$, and $WZ(D'_i)$ is a section of the pullback bundle of $WZ(\Gamma'_i)$ by the restriction map $r'_i : D'_i G \longrightarrow \Gamma_i G$. Similarly, for $i \in I_+$, $WZ(D'_i)$ defines a section of the pullback line bundle of $WZ(\Gamma_i)$ by r'_i, and $WZ(D_i)$ is a section of $r^*_i WZ(\Gamma'_i)$.

1 Let $\Sigma_1 \in \mathcal{M}_4$ and suppose that the compactified space $(\Sigma_1)^c$ is simply connected. that is. Σ_1 is a subset of S^4 deleted several discs $D_i; i \in I_-$ and $D'_i; i \in I_+$ with parametrized boundaries $\Gamma = \cup_{i\in I_-}\Gamma_i \cup \cup_{i\in I_+}\Gamma_i$. Let

$$\Phi_1 = \otimes_{i\in I_+}WZ(D_i) \otimes \otimes_{i\in I_-}WZ(D'_i). \qquad (3.7)$$

Φ_1 is a section of the pullback bundle of $WZ(\Gamma')$ by the restriction map

$$\left(\cup_{i\in I_-} D'_i \cup \cup_{i\in I_+} D_i\right) G \longrightarrow \left(\cup_{i\in I_-}\Gamma_i \cup \cup_{i\in I_+}\Gamma_i\right) G.$$

Then $WZ(\Sigma_1)$ is defined by the duality relation;

$$< WZ(\Sigma_1), \Phi_1 >= WZ(S^4) = 1. \qquad (3.8)$$

In fact, given $f \in \Sigma_1 G$, take $f_i \in D_i G$, $i \in I_+$, and $f'_i \in D'_i G$, $i \in I_-$, in such a way that $f|\Gamma_i = f_i|\Gamma_i$, $i \in I_+$, and $f|\Gamma_i = f'_i|\Gamma_i$, $i \in I_-$. Let $WZ(D_i)(f_i) = (f_i, u_i)$, $i \in I_+$, and $WZ(D'_j)(f'_j) = (f'_j, u'_j)$, $j \in I_-$. By the definition

$$u_i \in WZ(\Gamma'_i)_{r_i(f_i)}, \quad \text{and} \quad u'_j \in WZ(\Gamma'_j)_{r'_j(f'_j)}.$$

Then $\Phi_1((f_i)_{i \in I_+}, (f'_j)_{j \in I_-}) = ((f_i)_{i \in I_+}, (f'_j)_{j \in I_-}, \otimes_{i \in I_+} u_i \otimes \otimes_{j \in I_-} u'_j)$. There is a

$$v \in \otimes_{i \in I_+} WZ(\Gamma_i)_{r_i(f_i)} \otimes \otimes_{j \in I_-} WZ(\Gamma_j)_{r'_j(f'_j)} = WZ(\Gamma)_{r(f)}.$$

such that $< v, \otimes_{i \in I_+} u_i \otimes \otimes_{j \in I_-} u'_j >= 1$. The definition of $WZ(D_i)$ and $WZ(D'_i)$ imply that v is independent of $\{f_i, f'_i\}$, but depends only on f.

Thus $WZ(\Sigma_1)(f) = (f, v)$ is well defined as a section of the pullback bundle of $WZ(\Gamma)$ by $r : \Sigma_1 G \longrightarrow \Gamma G$.

2 Let $\Sigma_0 = S^3 \times [0, 1]$. We define

$$WZ(\Sigma_0) = 1_{WZ((S^3)') \otimes WZ(S^3)}. \tag{3.9}$$

Then we have

$$< WZ(\Sigma_0), WZ(D) \otimes WZ(D') >=< WZ(D), WZ(D') >= 1.$$

This is concordant with the definition in paragraph **1** .

3 We shall call a $\Sigma_1 \in \mathcal{M}_4$ described in **1** that is not of cylinder type a *basic component*. Any $\Sigma \in \mathcal{M}_4$ can be decomposed to a sum of several basic components that are patched together by their parametrized boundaries:

$$\Sigma = \cup_{k=1}^N \Sigma_k. \tag{3.10}$$

The incoming boundaries of Σ_k coincide respectively with the outgoing boundaries of Σ_{k-1} up to their orientations , that is, $\Gamma_i^{k-1} = (\Gamma_i^k)'$, and Σ is obtained by patching together these boundaries. Then there is a duality of $WZ(\Gamma_i^{k-1}) = (p_i^{k-1})^* WZ((S^3)')$ and $WZ(\Gamma_i^k) = (p_i^k)^* WZ(S^3)$. Using a suitable Morse function on Σ, we may suppose that the parametrized boundaries $\Gamma_i; i \in I_-$ of Σ are all contained in the boundary $\partial \Sigma_1$ and $\Gamma_i; i \in I_+$ are in $\partial \Sigma_N$. Then we define

$$WZ(\Sigma_2 \cup \Sigma_1) =< WZ(\Sigma_2), WZ(\Sigma_1) > . \tag{3.11}$$

Here $<,>$ is the natural pairing (contraction) between the line bundles $\otimes_{i \in I_-} WZ(\Gamma_i) \otimes \otimes_{j \in J_+^1} WZ(\Gamma_j)$ and $\otimes_{j \in J_-^2} WZ(\Gamma_j) \otimes \otimes_{k \in J_+^2} WZ(\Gamma_k)$. Here we have written $\partial \Sigma_1 = \cup_{j \in J_+^1} \Gamma_j \bigcup \cup_{i \in I_-} \Gamma_i$ and $\partial \Sigma_2 = \cup_{k \in J_+^2} \Gamma_k \bigcup \cup_{j \in J_-^2} \Gamma_j$, hence $WZ(\Sigma_2 \cup \Sigma_1)$ is a section of the pullback line bundle of $\otimes_{i \in I_-} WZ(\Gamma_i) \otimes \otimes_{k \in J_+^2} WZ(\Gamma_k)$ by the boundary restriction map

$$r : (\Sigma_2 \cup \Sigma_1) G \longrightarrow \cup_{i \in I_-} \Gamma_i G \cup \cup_{k \in J_+^2} \Gamma_k G,$$

see the explanation after **A3** of 1.2.

Lemma 3.1. *Let $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. Let their boundaries be*

$$\partial\Sigma_1 = \gamma_1 \cup \Gamma'_2 \cup \Gamma_3, \quad \partial\Sigma_2 = \gamma_2 \cup \Gamma'_3 \cup \Gamma_1 \quad \text{and} \quad \partial\Sigma_3 = \gamma_3 \cup \Gamma'_1 \cup \Gamma_2.$$

Then we have

$$<< WZ(\Sigma_1), WZ(\Sigma_2) >, WZ(\Sigma_3) >=< WZ(\Sigma_1), < WZ(\Sigma_2), WZ(\Sigma_3) >> \tag{3.12}$$

This is merely the problem of forming a tensor product of several line bundles, that is a commutative operation.

By virtue of this lemma we can form successively

$$\begin{aligned} WZ(\Sigma_k \cup \Sigma_{(k-1)} \cdots \cup \Sigma_1) \\ =< WZ(\Sigma_k), < WZ(\Sigma_{(k-1)}), \cdots, WZ(\Sigma_1) > \cdots > . \end{aligned} \tag{3.13}$$

This is independent of the order of partition and is also independent of how to decompose $\Sigma^{(k)} = \Sigma_k \cup \Sigma_{(k-1)} \cdots \cup \Sigma_1$, but depends only on $\Sigma^{(k)}$. Therefore

$$WZ(\Sigma) = WZ(\Sigma_N \cup \Sigma_{(N-1)} \cdots \cup \Sigma_1)$$

is well defined as a section of the pullback line bundle of $\otimes_{i \in I_-} WZ(\Gamma_i) \otimes \otimes_{i \in I_+} WZ(\Gamma_i)$ by the boundary restriction map.

From the construction $WZ(\Sigma)$ satisfies axiom **A3**.

Now let $\Sigma \in \mathcal{M}_4$ be compact without boundary. Let Σ_1 and Σ_2 be the basic components such that $\Sigma = \Sigma_1 \cup_\Gamma \Sigma_2$. Suppose that

$$\partial\Sigma_1 = \Gamma = \cup_{i \in I}\Gamma'_i, \quad \partial\Sigma_2 = \Gamma = \cup_{i \in I}\Gamma_i.$$

Then from the definition of $WZ(\Sigma_i)$, $i = 1, 2$, we see that:

$$\begin{aligned} WZ(\Sigma) &= < WZ(\Sigma_2), WZ(\Sigma_1) >=< \otimes_{i \in I} WZ(D'_i), \otimes_{i \in I} WZ(D_i) > \\ &= \sum_{i \in I} 1 . \end{aligned}$$

Thus we have the following

Proposition 3.2. *For any $\Sigma \in \mathcal{M}_4$ which is compact without boundary $WZ(\Sigma)$ is a positive integer.*

Proposition 3.3. *Let $\Sigma \in \mathcal{M}_4$ and let Σ^{ij} be obtained from Σ by identifying the boundaries $\Gamma_i, i \in I_-$, and $\Gamma_j, j \in I_+$, via $p_j \cdot (p_i)^{-1} : \Gamma_i \longrightarrow \Gamma_j$. Then*

$$WZ(\Sigma^{ij}) = Tr_{ij} WZ(\Sigma), \tag{3.14}$$

where Tr_{ij} are the trace maps (contraction) between $r^* WZ(\Gamma'_i)$ and $r^* WZ(\Gamma_j)$ in the tensor product $\otimes_{k \in I_-} WZ(\Gamma'_k) \otimes \otimes_{l \in I_+} WZ(\Gamma_l)$.

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Connections on $WZ(\Gamma_i)$ and $WZ(\Gamma'_i)$ are defined naturally as the induced connections by (3.1) and (3.2). Obviously $WZ(D_i)$ and $WZ(D'_i)$ are parallel with respect to these connections. By the formulas of definitions (3.3), (3.8) and (3.13) we have a naturally induced connection on $WZ(\Gamma)$ with respect to which $WZ(\Sigma)$ is parallel. Therefore axiom **A4** is verified.

Remark 3.1. Let $\Sigma \in \mathcal{M}_4$ and the boundary $\Gamma = \partial\Sigma$ be such that $\Gamma = \bigcup_{i\in I_+} \Gamma_i \cup \bigcup_{i\in I_-} \Gamma'_i$ with Γ_i a parametrized round S^3. Let $r_\pm$ denote respectively the restriction maps onto $\otimes_{i\in I_\pm}(\Gamma_i G)$. Then

$$WZ(\Sigma) : r_-^* \left(\otimes_{i\in I_-} WZ(\Gamma_i)\right) \longrightarrow r_+^* \left(\otimes_{i\in I_+} WZ(\Gamma_i)\right).$$

$WZ(\Sigma)(f)$ for $f \in \Sigma G$ is the higher-dimensional parallel transport along the "path" f, [19]. When $I_+ = \phi$ or $I_- = \phi$ we call $WZ(\Sigma)(f)$ the higher-dimensional holonomy along f.

3.3

To see that the functor WZ satisfies the axioms of WZW model it remains for us to verify the axiom **A5**, the Polyakov-Wiegmann formula on every $\Sigma \in \mathcal{M}_4$. We have already seen the Polyakov-Wiegmann formula on $S^4 G$, DG and $D'G$ in (2.17), (2.37) and (2.40) respectively.

Let $\Sigma \in \mathcal{M}_4$ with parametrized boundaries $\Gamma = \cup_{i\in I_-}\Gamma_i \cup \cup_{j\in I_+}\Gamma_j$. We shall use the same notation as in 3.1 and 3.2. Then the product on each pullback line bundle $r_i^* WZ(\Gamma_i)$, $(r')_i^* WZ(\Gamma_i)$, $r_i^* WZ(\Gamma'_i)$ and $(r')_i^* WZ(\Gamma'_i)$ is defined in an obvious manner, and the non-vanishing sections $WZ(D_i)$ and $WZ(D'_i)$ for $i \in I_\pm$ satisfy the respective Polyakov-Wiegmann formula

$$WZ(D_i)(fg) = WZ(D_i)(f) * WZ(D_i)(g), \qquad \text{etc..}$$

The products on the line bundle $S = \otimes_{j\in I_-}(r'_j)^* WZ(\Gamma_j) \otimes \otimes_{i\in I_+}(r_i)^* WZ(\Gamma_i)$ and on the line bundle $S^* = \otimes_{j\in I_-}(r'_j)^* WZ(\Gamma'_j) \otimes \otimes_{i\in I_+}(r_i)^* WZ(\Gamma'_i)$ are defined by tensoring the product on each $r_i^* WZ(\Gamma_i)$, etc.. We note also that the products are compatible with the duality:

$$< \alpha * \beta, \lambda * \mu > = < \alpha, \lambda > * < \beta, \mu >,$$

for α, $\beta \in S$ and λ, $\mu \in S^*$. Where the product in the right side is that in $WZ(\phi) \simeq \mathrm{C}$.

Now suppose that Σ is a subset of S^4 deleted several discs D_i, $i \in I_\pm$. Let $r : \Sigma G \longrightarrow \Gamma G$ be the restriction map. Then the product on $r^* WZ(\Gamma)$ is derived from the product on S. In fact, if we write $r(f) = (r_i(f_i); i \in I_+, r'_j; (f'_j), j \in I_-)$ as in the argument of 3.2, then $WZ(\Gamma)_{r(f)} = S_{r'_j(f'_j), r_i(f_i)}$,

so the product on S yields that on $r^* WZ(\Gamma)$, which is seen to be independent of the choice of $\{f_i, f'_j\}$.

Let $\Phi_1 = \otimes_{i \in I_+} WZ(D_i) \otimes \otimes_{j \in I_-} WZ(D'_j)$. Φ_1 is a section of the line bundle S^* and satisfies

$$\Phi_1(f'g') = \Phi_1(f') * \Phi_1(g'),$$

for $f', g' \in \otimes_{i \in I_+} D_i G \otimes \otimes_{i \in I_-} D'_i G$. Since the section $WZ(\Sigma)$ of $r^* WZ(\Gamma)$ was defined by the duality; $< WZ(\Sigma), \Phi_1 > = WZ(S^4)$, we have

$$\begin{aligned}
< WZ(\Sigma)(fg), \Phi_1(f'g') > &= WZ(S^4)(fg \vee f'g') \\
&= WZ(S^4)(f \vee f') * WZ(S^4)(g \vee g') = < WZ(\Sigma)(f), \Phi_1(f') > * < WZ(\Sigma)(g), \Phi_1(g') > \\
&= < WZ(\Sigma)(f) * WZ(\Sigma)(g), \Phi_1(f'g') >,
\end{aligned}$$

for any $f, g \in \Sigma G$ and for f' and g' that are extensions of f and g to $\cup_{i \in I_-} D'_i \cup \cup_{j \in I_+} D_j$ respectively. Therefore we have

$$WZ(\Sigma)(fg) = WZ(\Sigma) * WZ(\Sigma)(g),$$

for $f, g \in \Sigma G$.

Let $\Sigma = \Sigma_1 \cup_\Gamma \Sigma_2$. The product operations on $(r_i)^* WZ(\Gamma_i)$, $i = 1, 2$, are compatible with the contraction, in particular we have

$$\begin{aligned}
< WZ(\Sigma_1)(f_1) * WZ(\Sigma_1)(g_1) &\quad, \quad WZ(\Sigma_2)(f_2) * WZ(\Sigma_2)(g_2) > \\
&= WZ(\Sigma)(f) * WZ(\Sigma)(g), \quad (3.16)
\end{aligned}$$

where $f, g \in \Sigma G$ and $f_i = f|\Sigma_1$, $i = 1, 2$ etc.. For a general $\Sigma \in \mathcal{M}_4$ the formula follows from (3.14) and the definition of $WZ(\Sigma)$ in (3.12). Thus we have proven the following generalization of the Polyakov-Wiegmann formula.

Theorem 3.4.

$$WZ(\Sigma)(f) * WZ(\Sigma)(g) = WZ(\Sigma)(fg) \qquad (3.17)$$

for $f, g \in \Sigma G$.

4 Extensions of the group $\Omega_0^3 G$

It is a well known observation that the two-dimensional WZW action gives a geometric description of the central extension $\widehat{LG}$ of the loop group LG. The associated group cocycle yields a Lie algebra cocycle for the affine Kac-Moody algebra based on $Lie(G)$, [2, 7]. The total space of the $U(1)$-principal bundle $\widehat{LG}$ was described as the set of equivalence classes of pairs

$(f, c) \in D^2 G \times U(1)$, where D^2 is the 2-dimensional disc with boundary S^1. The equivalence relation was defined on the basis of Polyakov-Wiegmann formula [13], as it was so in our four-dimensional generalization treated in section 2.

Associated to the line bundle $WZ(S^3)$ there exists a $U(1)$-principal bundle over $\Omega_0^3 G$. However this bundle has not any natural group structure contrary to the case of the extension of loop group. Instead J. Mickelsson in [11] gave an extension of $\Omega_0^3 G$ by the abelian group $Map(\mathcal{A}_3, U(1))$, where $\mathcal{A}_3$ is the space of connections on S^3 . In the following we shall explain after [10] two extensions of $\Omega_0^3 G$ by the abelian group $Map(\mathcal{A}_3, U(1))$ that are in duality.

4.1

We consider the quotient space

$$\widehat{\Omega G} = D' G \times Map(\mathcal{A}_3, U(1))/ \sim', \qquad (4.1)$$

where $\sim'$ is the equivalence relation defined by

$$(f', \lambda) \sim' (g', \mu) \text{ if and only if } \begin{cases} f'|S^3 & = g'|S^3 \\ \mu(A) & = \lambda(A)\chi'(f', g') \end{cases} \quad \text{for any} A \in \mathcal{A}_3.$$
$$(4.2)$$

The projection $\pi : \widehat{\Omega G} \longrightarrow \Omega_0^3 G$ is defined by $\pi([f', \lambda]) = f'|S^3$. Then $\widehat{\Omega G}$ becomes a principal bundle over $\Omega_0^3 G$ with the structure group $Map(\mathcal{A}_3, U(1))$. Here the $U(1)$ valued transition function $\chi'(f', g')$ is considered as a constant function in $Map(\mathcal{A}_3, U(1))$.

The group structure of $\widehat{\Omega G}$ is given by the Mickelssons 2-cocycle (2.9) on D':

$$\gamma_{D'}(\cdot; f', g'), \qquad \text{for } f', g' \in D' G.$$

We note that, since it is the coboundary of

$$\frac{i}{24\pi^3} \int_{D'} \alpha_4(A; f'),$$

$\gamma_{D'}$ is in fact a cocycle. We define the product on $D' G \times Map(\mathcal{A}_3, U(1))$ by

$$(f', \lambda) * (g', \mu) = (f'g', \lambda(\cdot)\mu_{f'}(\cdot) \exp\{2\pi i \gamma_{D'}(A; f', g')\}), \qquad (4.3)$$

where

$$\mu_{f'}(A) = \mu((f'|S^3)^{-1} A (f'|S^3) + (f'|S^3)^{-1} d(f'|S^3)).$$

Then $D'G \times Map(\mathcal{A}_3, U(1))$ is endowed with a group structure and $\widehat{\Omega G}$ inherits it. The group $Map(\mathcal{A}_3, U(1))$ is embedded as a normal subgroup in $\widehat{\Omega G}$. Thus $\widehat{\Omega G}$ is an extension of $\Omega_0^3 G$ by the abelian group $Map(\mathcal{A}_3, U(1))$, [10, 11].

We have another extension of $\Omega_0^3 G$ by $Map(\mathcal{A}_3, U(1))$ if we consider

$$\widehat{\Omega' G} = DG \times Map(\mathcal{A}_3, U(1)))/\sim, \tag{4.4}$$

where the equivalence relation $\sim$ is defined by

$$(f, \lambda) \sim (g, \mu) \text{ if and only if } \begin{cases} f|S^3 &= g|S^3 \\ \mu(A) &= \lambda(A)\chi(f, g) \end{cases} \text{ for any } A \in \mathcal{A}_3. \tag{4.5}$$

The product on $\widehat{\Omega' G}$ is defined by the same way as above using the 2-cocycle $\gamma_D(A; f, g)$ of (2.9), and $\widehat{\Omega' G}$ becomes a extension of $\Omega_0^3 G$ by the abelian group $Map(\mathcal{A}_3, U(1))$.

The group $Map(\mathcal{A}_3, U(1))$ acts on C by $\lambda \cdot c = \lambda(0)c$. Then the associated line bundle to $\widehat{\Omega G}$ is $WZ(S^3)$ and that associated to $\widehat{\Omega' G}$ is $WZ((S^3)')$.

Remark 4.1. Consider the empty three manifold ϕ and look it as the boundary of S^4 . Then we may follow the above definition to have an extension of ϕG by $Map(\mathcal{A}_3, U(1))$. It becomes $\widehat{\phi G} = S^4 G \times Map(\mathcal{A}_3, U(1)))/\sim$, where

$$(F, \lambda) \sim (G, \mu) \text{ if and only if } \mu(A) = \exp\{2\pi i\omega(F, F^{-1}G)\}\lambda(A) \quad \text{for any } A.$$

Then, since $(F, \lambda) \sim (F, \lambda(0))$, it reduces to $\widehat{\phi G} = S^4 G \times U(1)/\sim$, that is, $\widehat{\phi G} \simeq U(1)$. The product in $\widehat{\phi G}$ may be defined by the same formula as in (4.3), but we have seen that it reduces to that of (2.16) because of the equality $\gamma_{S^4}(A; F, G) = \gamma(F, G)$, (2.10).

The duality between two extensions $\widehat{\Omega G}$ and $\widehat{\Omega' G}$ is given as follows. For $[f', \lambda] \in \widehat{\Omega G}$ and $[f, \alpha] \in \widehat{\Omega' G}$, we put

$$< [f', \lambda], ([f, \alpha] >= [f \vee f', \lambda(0)\alpha(0)], \tag{4.6}$$

where on the right hand side we used the product in $\widehat{\phi G} \simeq U(1)$. In fact, suppose that $(f', \lambda) \sim' (g', \mu)$ and $(f, \alpha) \sim (g, \beta)$. Then we have

$$\begin{aligned} \mu(A)\beta(A) &= \lambda(A)\alpha(A)\exp\{2\pi i\left[\gamma_{D'}(A; f', g') + \gamma_D(A; f, g)\right]\} \quad (4.7) \\ &= \lambda(A)\alpha(A)\exp\{2\pi i\gamma(f \vee f', g \vee g')\}. \quad (4.8) \end{aligned}$$

Here we used the relation (2.11).

The Lie algebra cocycle corresponding to the group cocycle γ_D is calculated in [11]. It is given by

$$
\begin{aligned}
c(A; X, Y) &= \frac{i}{12\pi^2} \int_D tr \, dA(dX dY + dY dX) \\
&= \frac{i}{12\pi^2} \int_{S^3} tr(A(dX dY + dY dX)). \quad (4.9)
\end{aligned}
$$

The Lie algebra cocycle corresponding to the group cocycle $\gamma_{D'}$ is given by

$$
\begin{aligned}
&\frac{i}{12\pi^2} \int_{D'} tr \, dA(dX dY + dY dX) \\
&= -\frac{i}{12\pi^2} \int_{S^3} tr(A(dX dY + dY dX)) = -c(A; X, Y) \quad (4.10)
\end{aligned}
$$

4.2   Remarks

1   The Euclidean action of a field $\varphi : \Sigma \longrightarrow G$ in WZW confomal field theory is defined as

$$
S_\Sigma(\varphi) = -\frac{ik}{12\pi^2} \int_\Sigma tr(d\varphi^{-1} \wedge *d\varphi) + C_\Sigma(\varphi). \quad (4.11)
$$

$S_\Sigma(\varphi)$ is invariant under a conformal change of metric and the second term $C_\Sigma(\varphi)$ is required to obtain a conformal invariance of the action. This was shown by K. Fujii in [6], and first noticed by E. Witten in [20] for the two-dimensional WZW model. The kinetic term in (4.11) is linear with respect to the multiplication of the fields;

$$
\int_\Sigma tr(d(fg)^{-1} \wedge *d(fg)) = \int_\Sigma tr(df^{-1} \wedge *df) + \int_\Sigma tr(dg^{-1} \wedge *dg) \quad (4.12)
$$

and does not affect the Polyakov-Wiegmann formula. Hence we prefered only to deal with the topological term $C_\Sigma(f)$, [3].

2   The argument in this paper will be valid also for $2n$-dimensional conformally flat manifolds with boundary if the Lie group $G = SU(N)$ is such that $N \geq n + 1$, in this case we have $\pi_{2n}(G) = 0$ and $\pi_{2n+1}(G) = \mathbf{Z}$. We have also the abelian extensions of $\Omega_0^{2n-1}G$ by $U(1)$. For that purpose we must have the Polyakov-Wiegmann formula for the action functional

$$
C_{2n+1}(f) = \frac{-i}{(2n-1)!(2\pi i)^{2(n-1)}} \int_{D^{2n+1}} tr(\tilde{g}^{-1} d\tilde{g})^{2n+1}, \quad g \in S^{2n}G.
$$

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See [6]. It seems that Polyakov-Wiegmann formula has not yet been proved for general n larger than 3.

3   Losev, Moore, Nekrasov and Shatashvili [9] discussed a four-dimensional WZW theory based on Kähler manifolds. Their Lagrangian is defined by

$$-\frac{1}{4\pi}\int_{\Sigma}\omega \wedge Tr(g^{-1}\partial g \wedge *g^{-1}\overline{\partial}g) + \frac{i}{12\pi}\int_{\Sigma\times[0,1]}\omega \wedge Tr(g^{-1}dg)^{3}.$$

The theory has the finiteness properties for the one-loop renormalization of the vacuum state. The authors studied the algebraic sector of their theory. The category of algebraic manifolds is not well behaved under contraction, hence their theory does not fit our axiomatic description.

4 S^4 is obtained by patching together two quarternion spaces and we have the conjugation $q \longrightarrow q^{-1}$ on it. Under the conjugation $WZ(S^4)$ is invariant but $WZ(D)$ and $WZ(D')$ will interchange. Since the conjugation inverts the orientation, $WZ(\Sigma)$ is invariant under the conjugation of Σ. We can convince ourselves of this fact if we follow the argument to define $WZ(\Sigma)$ for a $\Sigma \in \mathcal{M}_4$. This is the CPT invariance.

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