自己双対ヤン・ミルズ方程式と
トロイダル・リー代数

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References:

[K.-Ikeda-Takasaki]

"Hierarchy of $(2+1)$-dimensional nonlinear Schrödinger equation, self-dual Yang-Mills equation, and toroidal Lie algebras",

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Related works

[Ikeda-Takasaki] *Internat. Math. Res. Notices*, No. 7, 329-369 (2001)

[K.-Ohta] *J. Phys.* **A34**, 10585-10592 (2001)

§. Self-dual Yang Mills equation

$\boldsymbol{A}_u = \boldsymbol{A}_u(y, z, \bar{y}, \bar{z})$ $(u = y, z, \bar{y}, \bar{z})$: matrix-valued functions

Field strength : $\boldsymbol{F}_{uv}$ $(u, v = y, z, \bar{y}, \bar{z})$

$$\boldsymbol{F}_{uv} = \partial_u \boldsymbol{A}_v - \partial_v \boldsymbol{A}_u + [\boldsymbol{A}_u, \boldsymbol{A}_v].$$

Self-duality equation :

$$\boldsymbol{F}_{yz} = \boldsymbol{F}_{\bar{y}\bar{z}} = 0, \qquad \boldsymbol{F}_{y\bar{y}} + \boldsymbol{F}_{z\bar{z}} = 0$$

Invariant under the *gauge-transformation*,

$$\boldsymbol{A}_u \mapsto \tilde{\boldsymbol{A}}_u = \boldsymbol{G}^{-1}\boldsymbol{A}_u\boldsymbol{G} + \boldsymbol{G}^{-1}(\partial_u \boldsymbol{G}).$$

Yang's formulation

We can choose $\boldsymbol{G}$ and $\bar{\boldsymbol{G}}$ such that

$$\begin{cases} \partial_y \boldsymbol{G} = \boldsymbol{G}\boldsymbol{A}_y, \\ \partial_z \boldsymbol{G} = \boldsymbol{G}\boldsymbol{A}_z, \end{cases} \qquad \begin{cases} \partial_{\bar{y}} \bar{\boldsymbol{G}} = \bar{\boldsymbol{G}}\boldsymbol{A}_{\bar{y}}, \\ \partial_{\bar{z}} \bar{\boldsymbol{G}} = \bar{\boldsymbol{G}}\boldsymbol{A}_{\bar{z}}. \end{cases}$$

If we define the matrix $\boldsymbol{J}$ as $\boldsymbol{J} \stackrel{\text{def}}{=} \boldsymbol{G}\bar{\boldsymbol{G}}^{-1}$, the self-duality equation takes the form

$$\partial_{\bar{y}}\left(\boldsymbol{J}^{-1}\partial_y \boldsymbol{J}\right) + \partial_{\bar{z}}\left(\boldsymbol{J}^{-1}\partial_z \boldsymbol{J}\right) = 0.$$

SDYM as integrable system

- **Lax pair** (Belavin-Zakharov)

$$\begin{cases} \left(\frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial \bar{z}}\right) \boldsymbol{\Psi} &= \left(\boldsymbol{A}_y + \zeta \boldsymbol{A}_{\bar{z}}\right) \boldsymbol{\Psi} \\ \left(\frac{\partial}{\partial z} - \zeta \frac{\partial}{\partial \bar{y}}\right) \boldsymbol{\Psi} &= \left(\boldsymbol{A}_z - \zeta \boldsymbol{A}_{\bar{y}}\right) \boldsymbol{\Psi} \end{cases}$$

- **Hirota bilinear method** (Sasa-Ohta-Matsukidaira)

  Setting $\boldsymbol{J} = \frac{1}{f}\begin{pmatrix} 1 & -g \\ e & f^2 - eg \end{pmatrix}$, we introduce "$\tau$-functions"

  as follows:

$$e = \frac{\tau_{n+1,m-1}}{\tau_{n,m}}, \quad f = \frac{\tau_{n,m-1}}{\tau_{n,m}}, \quad g = \frac{\tau_{n-1,m-1}}{\tau_{n,m}},$$

  which satisfy Hirota-type equations,

$$\begin{aligned} D_y \tau_{n,m} \cdot \tau_{n+1,m-1} + D_{\bar{z}} \tau_{n+1,m} \cdot \tau_{n,m-1} &= 0 \\ D_z \tau_{n,m} \cdot \tau_{n+1,m-1} - D_{\bar{y}} \tau_{n+1,m} \cdot \tau_{n,m-1} &= 0 \\ \tau_{n,m}^2 + \tau_{n,m+1}\tau_{n,m-1} - \tau_{n+1,m}\tau_{n-1,m} &= 0 \end{aligned}$$

- **Twistor approach** (Atiyah-Ward)

- **Riemann-Hilbert problem** (Ueno-Nakamura)

- **Darboux transformation** (Nimmo-Gilson-Ohta)

Ward's conjecture

Ward (1985) has conjectured that:

> *"many (and pahaps all?) of the ordinary and partial differential equations that are regarded as being integrable or solvable may be obtained from the self-dual gauge field equations (or its generalizations) by reduction".*

(2+1)-dimension

- 2-dim. Toda equation $\rightarrow \begin{cases} \text{KP equation} \\ \text{Davey-Stewartson equaion} \end{cases}$
- (2+1)-dimensional chiral field equation (Ward)

(1+1)-dimension

- KdV equation, NLS equation, sine-Gordon equation
- Chiral field equation, Ernst equations

1-dimension

- Euler-Arnold top
- Chazy equation
- Painlevé equaions

SDYM hierarchy

(Nakamura (1987), Takasaki (1990), Ablowitz (1991))

The Lax pair for the SDYM equation

$$\begin{cases} \left(\frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial \bar{z}}\right) \boldsymbol{\Psi} &= \left(\boldsymbol{A}_y + \zeta \boldsymbol{A}_{\bar{z}}\right) \boldsymbol{\Psi} \\ \left(\frac{\partial}{\partial z} - \zeta \frac{\partial}{\partial \bar{y}}\right) \boldsymbol{\Psi} &= \left(\boldsymbol{A}_z - \zeta \boldsymbol{A}_{\bar{y}}\right) \boldsymbol{\Psi} \end{cases}$$

$$\Downarrow$$

The Lax pair for the SDYM **hierarchy**

$$\begin{cases} \left(\frac{\partial}{\partial y_n} + \zeta^n \frac{\partial}{\partial \bar{z}}\right) \boldsymbol{\Psi} &= \left(\boldsymbol{A}_y^{(n)} + \zeta^n \boldsymbol{A}_{\bar{z}}\right) \boldsymbol{\Psi} \\ \left(\frac{\partial}{\partial z_n} - \zeta^n \frac{\partial}{\partial \bar{y}}\right) \boldsymbol{\Psi} &= \left(\boldsymbol{A}_z^{(n)} - \zeta^n \boldsymbol{A}_{\bar{y}}\right) \boldsymbol{\Psi} \end{cases}$$

However, it is *not* straightforward to introduce "τ-functions" using these linear equations.

(Cf. Case of the KP hierarchy

Linear equations for $\boldsymbol{\Psi}$ $\rightarrow$ "τ-functions")

What is the meaning of the work of Sasa-Ohta-Matsukidaira?

An Idea:

> Introduce SDYM-type time-evolutions to the Sato Grassmannian

- 2 component KP hierarchy → **reduction** → NLS hierarchy

$$\boldsymbol{W} \stackrel{\text{def}}{=} \boldsymbol{E} + \sum_{n=1}^{\infty} \boldsymbol{w}_n(x)\partial_x^{-n} \quad \text{(PsDO with matrix coefficients)}$$

We impose the condition $\boxed{\boldsymbol{W}\partial_x \boldsymbol{W}^{-1} = \boldsymbol{E}\partial_x}$

Dependence on $\boldsymbol{x} = (x_1, x_2, \dots)$:

$$\frac{\partial \boldsymbol{W}}{\partial x_n} = \boldsymbol{B}_n \boldsymbol{W} - \boldsymbol{W}\boldsymbol{P}\partial_x^n, \quad \boldsymbol{B}_n = \left[\boldsymbol{W}\boldsymbol{P}\partial_x^n \boldsymbol{W}^{-1}\right]_{\geq 0}$$

- Hierarchy of gauge field equations

Dependence on $\boldsymbol{y} = (y_0, y_1, y_2, \dots)$:

$$\frac{\partial \boldsymbol{W}}{\partial y_n} = \boldsymbol{C}_n \boldsymbol{W} + \frac{\partial \boldsymbol{W}}{\partial y_0}\partial_x^n, \quad \boldsymbol{C}_n = -\left[\frac{\partial \boldsymbol{W}}{\partial y_0}\partial_x^n \boldsymbol{W}^{-1}\right]_{\geq 0}$$

Dependence on $\boldsymbol{z} = (z_0, z_1, z_2, \dots)$:

$$\frac{\partial \boldsymbol{W}}{\partial z_n} = \overline{\boldsymbol{C}}_n \boldsymbol{W} + \frac{\partial \boldsymbol{W}}{\partial z_0}\partial_x^n, \quad \overline{\boldsymbol{C}}_n = -\left[\frac{\partial \boldsymbol{W}}{\partial z_0}\partial_x^n \boldsymbol{W}^{-1}\right]_{\geq 0}$$

This hierarchy contains several interesting examples:

Example 1 : (2+1)-dim. NLS equation

(Bogoyavlensky (1990), Schiff (1992), Strachan (1993))

Consider the time evolutions w.r.t. x_1 and y_1.

$$
\begin{aligned}
\frac{\partial \boldsymbol{W}}{\partial x_1} &= \boldsymbol{B}_1 \boldsymbol{W} - \boldsymbol{W}\boldsymbol{P}\partial_x, \qquad \boldsymbol{B}_1 = \boldsymbol{P}\partial_x + \boldsymbol{w}_1\boldsymbol{P} - \boldsymbol{P}\boldsymbol{w}_1, \\
\frac{\partial \boldsymbol{W}}{\partial y_1} &= \boldsymbol{C}_1 \boldsymbol{W} + \frac{\partial \boldsymbol{W}}{\partial y_0}\partial_x, \qquad \boldsymbol{C}_1 = -\frac{\partial \boldsymbol{w}_1}{\partial y_0}.
\end{aligned}
$$

If we define (formal) Baker-Akhiezer function as

$$
\boldsymbol{\Psi} = \boldsymbol{W}\boldsymbol{\Psi}_0, \qquad \frac{\partial \boldsymbol{\Psi}_0}{\partial x} = \lambda \boldsymbol{\Psi}_0, \qquad \frac{\partial \boldsymbol{\Psi}_0}{\partial y_1} = \lambda \frac{\partial \boldsymbol{\Psi}_0}{\partial y_0},
$$

then we have

$$
\begin{cases}
\dfrac{\partial \boldsymbol{\Psi}}{\partial x_1} = (\boldsymbol{P}\lambda + \boldsymbol{w}_1\boldsymbol{P} - \boldsymbol{P}\boldsymbol{w}_1)\boldsymbol{\Psi}, \\
\dfrac{\partial \boldsymbol{\Psi}}{\partial y_1} = \left(\lambda \dfrac{\partial}{\partial y_0} - \dfrac{\partial \boldsymbol{w}_1}{\partial y_0}\right)\boldsymbol{\Psi}.
\end{cases}
$$

$$\Downarrow$$

$$
\boldsymbol{w_1} = \begin{pmatrix} w_1 & u \\ -u^* & w_2 \end{pmatrix}, \qquad X = \mathrm{i}x_1, \quad Y = y_0, \quad T = y_1
$$

$$\Downarrow$$

$$
\mathrm{i}u_T + u_{XY} + 2u\int^X (|u|^2)_Y \mathrm{d}X = 0
$$

Example 2 : (2+1)-dim. chiral field (Ward (1988))

Consider the time evolutions w.r.t. y_1 and y_{-1}.

$$\begin{cases} \dfrac{\partial \boldsymbol{\Psi}}{\partial y_1} = \left(\lambda \dfrac{\partial}{\partial y_0} - \dfrac{\partial \boldsymbol{w}_1}{\partial y_0}\right) \boldsymbol{\Psi}, \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial y_{-1}} = \lambda^{-1} \left(\dfrac{\partial}{\partial y_0} - \dfrac{\partial \tilde{\boldsymbol{w}}_0}{\partial y_0} \tilde{\boldsymbol{w}}_0^{-1}\right) \boldsymbol{\Psi}. \end{cases}$$

$$\Downarrow$$

$$\boxed{\begin{array}{c} A \stackrel{\text{def}}{=} \dfrac{\partial \boldsymbol{w}_1}{\partial y_0}, \qquad B \stackrel{\text{def}}{=} -\dfrac{\partial \tilde{\boldsymbol{w}}_0}{\partial y_0} \tilde{\boldsymbol{w}}_0^{-1} \\ \partial_{y_0} \to \partial_x, \quad \partial_{y_1} \to \partial_t + \partial_y, \quad \partial_{y_{-1}} \to \partial_t - \partial_y \end{array}}$$

$$\Downarrow$$

$$\begin{cases} (\partial_t + \partial_y)\psi = (\lambda \partial_x - A)\psi, \\ (\partial_t - \partial_y)\psi = \left(\dfrac{1}{\lambda}\partial_x + \dfrac{B}{\lambda}\right)\psi \end{cases}$$

$$\Downarrow$$

$$\boxed{A = \boldsymbol{J}^{-1}(\boldsymbol{J}_t + \boldsymbol{J}_y), \qquad B = \boldsymbol{J}^{-1}\boldsymbol{J}_x}$$

$$\Downarrow$$

$$\partial_t(\boldsymbol{J}^{-1}\boldsymbol{J}_t) - \partial_x(\boldsymbol{J}^{-1}\boldsymbol{J}_x) - \partial_y(\boldsymbol{J}^{-1}\boldsymbol{J}_y) + [\boldsymbol{J}^{-1}\boldsymbol{J}_y, \boldsymbol{J}^{-1}\boldsymbol{J}_t] = 0,$$

§. Lie algebraic aspects

Realization of $\widehat{\mathfrak{sl}}_2 \;\Rightarrow\;$ Soliton equations

[Date-Jimbo-Kashiwara-Miwa, 1981]

1-component KP		**2-component KP**
KP eq.	$\leftarrow \; \widehat{\mathfrak{gl}}_\infty \; \rightarrow$	DS eq.
	$\cup$	
KdV eq.	$\leftarrow \; \widehat{\mathfrak{sl}}_2 \; \rightarrow$	NLS eq.

NLS : **2-component fermions** $\psi_j^{(\alpha)}, \psi_j^{(\alpha)*}$ $(j \in \mathbb{Z}, \alpha = 1, 2)$

$$[\psi_i^{(\alpha)}, \psi_j^{(\beta)*}]_+ = \delta_{ij}\delta_{\alpha\beta}, \quad [\psi_i^{(\alpha)}, \psi_j^{(\beta)}]_+ = [\psi_i^{(\alpha)*}, \psi_j^{(\beta)*}]_+ = 0$$

$$\left[\widehat{\mathfrak{sl}}_2, \sum_{\alpha=1,2}\sum_{j\in\mathbb{Z}} \psi_j^{(\alpha)} \otimes \psi_j^{(\alpha)*}\right] = 0 \quad \rightarrow \quad \text{“Bosonization”}$$

$\rightarrow$ Bilinear Identity for NLS

“Bosonization”

$$H_n \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \psi_j \psi_{n+j}^* \;\Rightarrow\; [H_m, H_n] = m\delta_{m+n,0}$$

$$\left(H_n \sim \frac{\partial}{\partial x_n}, \quad H_{-n} \sim n x_n \right)$$

$\mathfrak{sl}_2^{\mathrm{tor}}$: toroidal $\mathfrak{sl}_2$ algebra

$$
\begin{array}{ccc}
\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}] & \Rrightarrow & \text{affine Lie algebras} \\
& \text{central extension} & \cap \\
\mathfrak{g}\otimes\mathbb{C}[s,s^{-1},t,t^{-1}] & \Rrightarrow & \textbf{2-toroidal Lie algebras}
\end{array}
$$

$$\boxed{\mathfrak{sl}_2^{\mathrm{tor}} = \mathfrak{sl}_2\otimes\mathbb{C}[s,s^{-1},t,t^{-1}]\oplus\Omega_A/\mathrm{d}A}$$

$$A=\mathbb{C}[s,s^{-1},t,t^{-1}],\quad \Omega_A=A\mathrm{d}s\oplus A\mathrm{d}t$$

Canonical projection $\overline{\cdot}$

$$
\begin{array}{ccccc}
\overline{\cdot} & : & \Omega_A & \to & \Omega_A/\mathrm{d}A \\
& & \Cup & & \Cup \\
& & f\mathrm{d}g & \mapsto & \overline{f\mathrm{d}g}
\end{array}
$$

Relations: • $[x\otimes a, y\otimes b]=[x,y]\otimes ab+(x|y)\overline{(\mathrm{d}a)b}$

$(\,x,y\in\mathfrak{sl}_2,\ a,b\in A,\ (x|y)$: Killing form $)$

• $[\mathfrak{sl}_2^{\mathrm{tor}},\Omega_A/\mathrm{d}A]=0$

Representation theory of $\mathfrak{sl}_2^{\mathrm{tor}}$ $\rightarrow$ Little is known so far.

There is a simple class of representations of toroidal Lie algebras:

(V, π) : a representaion of $\widehat{\mathfrak{g}}$ (affine Lie algebra)

$X_m(z)$, $K_m^s(z)$, $K_m^t(z)$: Operators acting on $V \otimes \mathbb{C}[y, \mathrm{e}^{\pm y_0}]$

$$\begin{aligned}
X_m(z) &\stackrel{\mathrm{def}}{=} X^{\pi}(z) \otimes V_m(z), \\
K_m^s(z) &\stackrel{\mathrm{def}}{=} c \cdot \mathrm{id}_V \otimes V_m(z), \\
K_m^t(z) &\stackrel{\mathrm{def}}{=} c \cdot \mathrm{id}_V \otimes \varphi(z) V_m(z), \\
\varphi(z) &\stackrel{\mathrm{def}}{=} \sum_{n \in \mathbb{Z}} n y_n z^{n-1}, \\
V_m(z) &\stackrel{\mathrm{def}}{=} \exp\left[m \sum_{n \in \mathbb{Z}} y_n z^n \right].
\end{aligned}$$

<u>Proposition</u>

$X_m(z)$, $K_m^s(z)$, $K_m^t(z)$ give a representation of $\mathfrak{g}^{\mathrm{tor}}$:

$$\begin{aligned}
X_m(z) &= \sum_{n \in \mathbb{Z}} (X \otimes s^n t^m) \cdot z^{-n-1}, \\
K_m^s(z) &= \sum_{n \in \mathbb{Z}} \overline{s^{n-1} t^m \, \mathrm{d}s} \cdot z^{-n}, \\
K_m^{t_k}(z) &= \sum_{n \in \mathbb{Z}} \overline{s^n t^{m-1} \, \mathrm{d}t} \cdot z^{-n-1}.
\end{aligned}$$

NLS-case [K.-Ikeda-Takasaki, 2001]

$$\left.\begin{array}{l} \psi^{(1)}(p)\psi^{(2)*}(p), \quad \psi^{(2)}(p)\psi^{(1)*}(p), \\ :\psi^{(1)}(p)\psi^{(1)*}(p): - :\psi^{(2)}(p)\psi^{(2)*}(p): \end{array}\right\} \quad \rightarrow \quad \text{realization of } \widehat{\mathfrak{sl}}_2$$

$\Downarrow$ **"toroidalization"**

$$\left.\begin{array}{l} \psi^{(1)}(p)\psi^{(2)*}(p)V_n(y,p), \quad \psi^{(2)}(p)\psi^{(1)*}(p)V_n(y,p), \\ \{:\psi^{(1)}(p)\psi^{(1)*}(p): - :\psi^{(2)}(p)\psi^{(2)*}(p):\}V_n(y,p), \end{array}\right\} \quad \rightarrow \quad \mathfrak{sl}_2^{\mathrm{tor}}$$

$$\left(V_n(y,p) = \exp\left(n \sum_{j=0}^{\infty} p^j y_j \right) \right)$$

"Casimir-like" operator

$$\Omega^{\mathrm{tor}} \stackrel{\mathrm{def}}{=} \sum_{m\in\mathbb{Z}} \sum_{\alpha=1,2} \oint \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\lambda} \psi^{(\alpha)}(\lambda)V_m(y,\lambda) \otimes \psi^{(\alpha)*}(\lambda)V_{-m}(y',\lambda)$$

Properties:

- $\left[\mathfrak{sl}_2^{\mathrm{tor}},\ \Omega^{\mathrm{tor}}\right] = 0$
- $\Omega^{\mathrm{tor}}|s_1,s_2\rangle \otimes |s_1,s_2\rangle = 0$

$\Downarrow$

Use "Boson-Fermion correspondence" and "Billig's Lemma"

$\Downarrow$

Bilinear Identity for (2+1)-dimensional NLS hierarchy

Lemma [Billig, 1999]

Let $P(n) = \sum_{j\geq 0} n^j P_j$, where P_j are differential operators that may not depend on z. If

$$\sum_{n\in\mathbb{Z}} z^n P(n) f(z) = 0$$

for some function $f(z)$, then

$$P(\epsilon - z\partial_z) f(z)|_{z=1} = 0$$

as a polynomial in ϵ.

Proof.

$$\begin{aligned}
0 &= \sum_{n\in\mathbb{Z}} z^n P(n) g(z) \;=\; \sum_{j\geq 0}\sum_{n\in\mathbb{Z}} \underbrace{n^j z^n}_{(z\partial_z)^j z^n} P_j g(z) \\
&= \sum_{j\geq 0} (z\partial_z)^j \delta(z) P_j g(z) \;=\; \sum_{j\geq 0} \delta^{(j)}(z) P_j g(z) \\
&= \sum_{j\geq 0}\sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} \delta^{(k)}(z) (P_j g)^{(j-k)}(z=1) \\
&= \sum_{k,l\geq 0} (-1)^l \frac{(k+l)!}{k!\,l!} \delta^{(k)}(z) (P_{k+l} g)^{(l)}(z=1) \\
\Rightarrow & \sum_{l=0}^{\infty} \frac{(k+l)!}{k!\,l!} (P_{k+l} g)^{(l)}(z=1) \;=\; 0
\end{aligned}$$

Introduce a formal parameter ϵ, we can rewrite as follows:

$$\begin{aligned}
0 &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} (-1)^l \epsilon^k \frac{(k+l)!}{k!l!} (P_{k+l} g)^{(l)}(z=1) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} (-1)^{n-k} \epsilon^k \frac{n!}{k!\,(n-k)!} (P_{k+l} g)^{(l)}(z=1) \\
&= \sum_{n=0}^{\infty} (\epsilon - z\partial_z)^n (P_n g)(z=1) \\
&= (P(\epsilon - z\partial_z) g)(z=1) \qquad \square
\end{aligned}$$

Bilinear Identity

$$(-1)^{s_2'+s_2''}\oint \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\lambda}\lambda^{s_1'-s_1''+k-1}\mathrm{e}^{\xi(-2a^{(1)},\lambda)}\mathrm{e}^{\xi(\tilde{D}^{(1)},\lambda^{-1})}\exp\left[\sum_{\alpha=1,2}\sum_{j=1}^{\infty}a_j^{(\alpha)}D_j^{(\alpha)}\right]$$

$$\times\ \exp\left[-\eta(\check{b},\lambda)D_{y_0}+\sum_{j=1}^{\infty}b_jD_{y_j}\right]\tau_{s_1,s_2}^{s_1''+1,s_2''}\cdot\tau_{s_1+l_1,s_2+l_2}^{s_1'-1,s_2'}$$

$$+\oint \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\lambda}\lambda^{s_2'-s_2''+k-1}\mathrm{e}^{\xi(-2a^{(2)},\lambda)}\mathrm{e}^{\xi(\tilde{D}^{(2)},\lambda^{-1})}\exp\left[\sum_{\alpha=1,2}\sum_{j=1}^{\infty}a_j^{(\alpha)}D_j^{(\alpha)}\right]$$

$$\times\ \exp\left[-\eta(\check{b},\lambda)D_{y_0}+\sum_{j=1}^{\infty}b_jD_{y_j}\right]\tau_{s_1,s_2}^{s_1'',s_2''+1}\cdot\tau_{s_1+l_1,s_2+l_2}^{s_1',s_2'-1}\ =\ 0$$

$$(\tau_{s_1,s_2}^{s_1',s_2'}=\langle s_1',s_2'|\mathrm{e}^{H(x^{(1)},x^{(2)})}g|s_1,s_2\rangle,\ \ \xi(x,\lambda)=\sum_{j=1}^{\infty}x_j\lambda^j,\ \ \eta(\check{b},\lambda)=\sum_{j=1}^{\infty}b_j\lambda^j)$$

($D_j^{(\alpha)}$: Hirota's bilinear operators)

Examples of Hirota-type equations:

$$\begin{cases} D_{x_1}^2\tau_{0,0}^{0,0}\cdot\tau_{0,0}^{0,0}+2\tau_{0,0}^{1,-1}\tau_{0,0}^{-1,1}\ =\ 0, \\ (D_{y_1}+D_{x_1}D_{y_0})\tau_{0,0}^{0,0}\cdot\tau_{0,0}^{1,-1}\ =\ 0, \\ (D_{y_1}+D_{x_1}D_{y_0})\tau_{0,0}^{-1,1}\cdot\tau_{0,0}^{0,0}\ =\ 0 \end{cases}$$

(Same equations as [Strachan], [Sasa-Ohta-Matsukidaira])

$$\Downarrow$$

$$u=\frac{\tau_{0,0}^{1,-1}}{\tau_{0,0}^{0,0}},\ u^*=\frac{\tau_{0,0}^{-1,1}}{\tau_{0,0}^{0,0}},\ X=\mathrm{i}x_1,\ Y=y_0,\ T=y_1$$

$$\Downarrow$$

$$\mathrm{i}u_T+u_{XY}+2u\int^X(|u|^2)_Y\mathrm{d}X\ =\ 0$$

((2+1)-dimensional NLS equation)

Equations from different sectors

$$\Omega^{\mathrm{tor}}\left(|s_1, s_2\rangle^{\mathrm{tor}} \otimes |s_1+1, s_2+1\rangle^{\mathrm{tor}}\right)$$

$$= \sum_{m\in\mathbb{Z}} \Big\{ (|s_1+1, s_2\rangle \otimes \mathrm{e}^{my_0}) \otimes \left(|s_1, s_2+1\rangle \otimes \mathrm{e}^{-my'_0}\right)$$

$$- (|s_1, s_2+1\rangle \otimes \mathrm{e}^{my_0}) \otimes \left(|s_1+1, s_2\rangle \otimes \mathrm{e}^{-my'_0}\right) \Big\}$$

$$\Downarrow$$

Examples of Hirota-type equations:

$$(\tau^{s_1,s_2}_{s_2,s_1})^2 + \tau^{s_1+1,s_2}_{s_2+1,s_1}\tau^{s_1,s_2+1}_{s_2,s_1+1} - \tau^{s_1+1,s_2}_{s_2,s_1+1}\tau^{s_1,s_2+1}_{s_2+1,s_1} = 0,$$

$$D_{y_0}\tau^{s_1+1,s_2-1}_{s_2,s_1} \cdot \tau^{s_1,s_2}_{s_2,s_1} = D_{y_1}\tau^{s_1+1,s_2}_{s_2,s_1+1} \cdot \tau^{s_1+1,s_2}_{s_2+1,s_1},$$

$$D_{y_0}\tau^{s_1-1,s_2+1}_{s_2,s_1} \cdot \tau^{s_1,s_2}_{s_2,s_1} = D_{y_1}\tau^{s_1,s_2+1}_{s_2,s_1+1} \cdot \tau^{s_1,s_2+1}_{s_2+1,s_1},$$

Setting

$$\boldsymbol{J} = \frac{1}{f}\begin{pmatrix} 1 & -g \\ e & f^2 - eg \end{pmatrix}, \quad e = \frac{\tau^{0,0}_{1,-1}}{\tau^{0,0}_{0,0}}, \quad f = \mathrm{i}\frac{\tau^{0,1}_{1,0}}{\tau^{0,0}_{0,0}}, \quad g = \frac{\tau^{-1,1}_{0,0}}{\tau^{0,0}_{0,0}}.$$

$$\bar{y} = y_0, \quad z = y_1, \quad \bar{z} = z_0, \quad y = -z_1,$$

($\{z_j\ (j = 0, 1, \ldots)\}$ play the same role as $\{y_j\}$)

$$\Downarrow$$

$$\frac{\partial}{\partial\bar{y}}\left(\boldsymbol{J}^{-1}\frac{\partial \boldsymbol{J}}{\partial y}\right) + \frac{\partial}{\partial\bar{z}}\left(\boldsymbol{J}^{-1}\frac{\partial \boldsymbol{J}}{\partial z}\right) = 0$$

(Self-duality equation in Yang's R-gauge)

Summary

Self-dual Yang-Mills ((2+2)-dimension)	**3-toroidal** $\mathfrak{sl}_3^{\mathrm{tor}}$
$\downarrow$	
(2+1)-dim. NLS KdV Chiral Field	**2-toroidal** $\mathfrak{sl}_2^{\mathrm{tor}}$
$\downarrow$	
(1+1)-dim. soliton eqs. NLS KdV Chiral Field	$\widehat{\mathfrak{sl}}_2 \subset \widehat{\mathfrak{gl}}_\infty$ $\longleftrightarrow$ Davey-Stewartson eq. $\longleftrightarrow$ KP eq. $\longleftrightarrow$ 2-dim. Toda eq.