Geometry of OSp

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§ Supermatrices and Linear Supergroups

$$GL(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \ {}^{\exists}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \quad a, b, c, d \in K \right\}$$

K : 体 → 可換代数 → 超代数

超代数 (superalgebras)

(i) R : 可換環

A : R-alg.

(ii) $A = A_0 \oplus A_1$ as R-module

$$\left.\begin{aligned} a \in A_0 &\overset{\text{def}}{\Longleftrightarrow} \hat{a} = 0 \\ a \in A_1 &\overset{\text{def}}{\Longleftrightarrow} \hat{a} = 1 \end{aligned}\right\} \ \hat{a} \in \mathbb{Z}/2\mathbb{Z}$$

(iii) $\widehat{a \cdot b} = \hat{a} + \hat{b}$ in $\mathbb{Z}/2\mathbb{Z}$

(iv)' supercommutativity

$$a \cdot b = (-1)^{\hat{a}\hat{b}} b \cdot a$$

$a, b \in A$

$a = a_0 \oplus a_1 \quad b = b_0 \oplus b_1$

$(a_0, b_0 \in A_0 \quad a_1, b_1 \in A_1)$

$$
\begin{aligned}
ab &= (a_0 \oplus a_1)(b_0 \oplus b_1) \\
&= (a_0 b_0 + a_1 b_1) \oplus (a_0 b_1 + a_1 b_0) \\
&= (b_0 a_0 - b_1 a_1) \oplus (b_1 a_0 + b_0 a_1)
\end{aligned}
$$

Ex

- Grassmann ⋯ supercommutative
- Clifford ⋯ non-supercommutative

$$
\begin{array}{ccc}
A & \xrightarrow{\pi} & A/(A_1) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
a & \longmapsto & a_{red}
\end{array}
$$

Ex

$A = \Lambda \mathbb{R}^m \quad \pi(A) \cong \mathbb{R}$

$v = v_0 + v_1 e_1 + \cdots + v_m e_m + \cdots + v_{m1} e_1 \wedge \cdots \wedge e_m$

$v_{red} = v_0$

- Supermatrices

$$X = \begin{array}{c c} & \begin{array}{cc} r & s \end{array} \\ \begin{array}{c} p \\ q \end{array} & \left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right) \end{array} \qquad \begin{array}{l} X_{11}, X_{22} \cdots \text{even} \\ X_{12}, X_{21} \cdots \text{odd} \end{array}$$

$$\cap$$

$$\mathrm{Mat}_{p|q \times r|s}(A)$$

- $X = (x_{ij})_{\substack{1 \leq i \leq m+n \\ 1 \leq j \leq p+q}} \in \mathrm{Mat}_{m|n \times p|q}(A)$

$X_{red} := ((x_{ij})_{red})$

<u>Lemma</u>

Xが可逆 $\Longleftrightarrow$ X_{red} が可逆

- Berezinian

$$\mathrm{Ber}\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} := \frac{\det(X_{11} - X_{12}X_{22}^{-1}X_{21})}{\det(X_{22})}$$

<u>Prop</u>

Ber : {可逆超行列} $\longrightarrow$ {可換代数}

準同型

Re

Berezin 積分の積分要素について

$$dx_1 \cdots dx_{m+n} = \mathrm{Ber}\left(\left(\frac{\partial x_i}{\partial y_j}\right)\right) dy_1 \cdots dy_{m+n}$$

- algebraic group … scheme + group

algebraic supergroup

… superscheme + group

Leites (1974) … superscheme の定式化

Brundan – Kleshchev (2003)

functor を用いた superscheme の定式化

- Lie superalgebra

(i) $V = V_0 \oplus V_1$ $\quad v \in V_0 \Leftrightarrow: \hat{v} = 0 \quad v \in V_1 \Leftrightarrow: \hat{v} = 1$

$\hat{v} \in \mathbb{Z}/2\mathbb{Z}$

(ii) $[u, v] = -(-1)^{\hat{u}\hat{v}} [v, u]$

(iii) $[u, [v, w]] + (-1)^{(\hat{u}+\hat{v})\hat{w}} [w, [u, v]]$

$+ (-1)^{\hat{u}(\hat{v}+\hat{w})} [v, [w, u]] = 0$

- G: compact Lie group $\mathfrak{g} := \mathrm{Lie}(G)$

  M: manifold

  $\alpha : G \curvearrowright M$ action

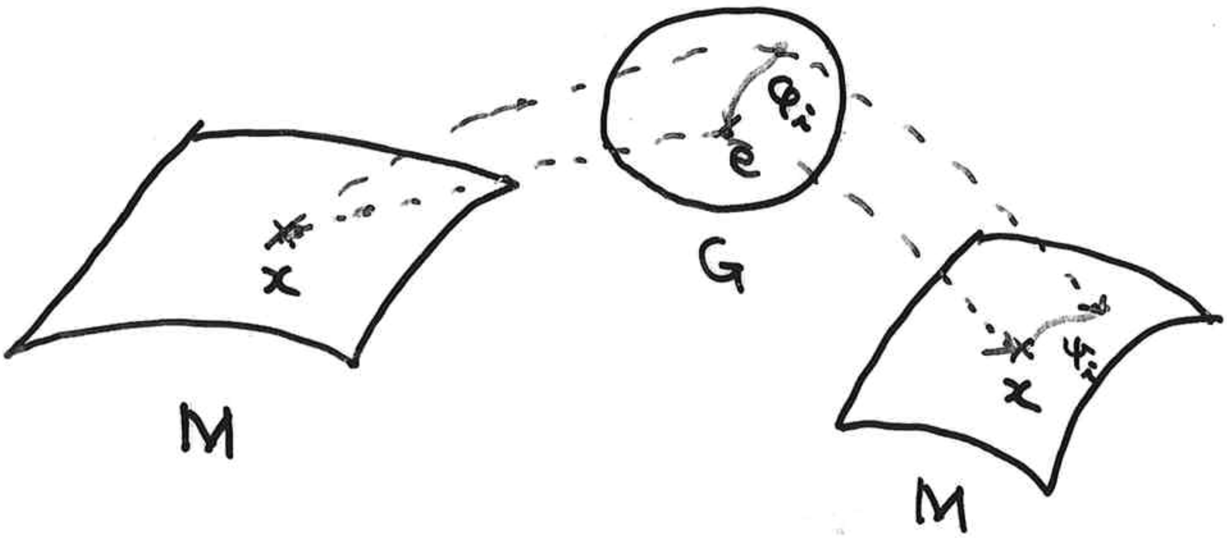
  $(e_1, \cdots, e_n)$ a basis of $\mathfrak{g}$

  $\mathfrak{g} = \{$inv. vec. fields on $G\}$

  $(\varphi_i)_t$: flow of e_i

  $(\psi_i)_t : M \longrightarrow M$

  $$(\psi_i)_t(x) := \alpha_{(\varphi_i)_t(e)}(x) \qquad (x \in M)$$



$v_1, \cdots, v_n$: v.f. on M corresponding to $\psi_1, \cdots, \psi_n$

$\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$

$\mathfrak{g}_0 := \{L_{v_1}, \cdots, L_{v_n}\}$ $\quad$ $\mathfrak{g}_{+1} := \{\iota_{v_1}, \cdots, \iota_{v_n}\}$

$\mathfrak{g}_{-1} := \{d\}$

$u, v \in \hat{\mathfrak{g}}$

$[u, v] := u \circ v - (-1)^{\hat{u}\hat{v}} v \circ u$

$\hat{\mathfrak{g}}$ Lie superalgebra $([\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j})$

$\hat{\mathfrak{g}} \curvearrowright \Omega(M)$ representation

$\downarrow$ unitary trick

$G^* \curvearrowright \Omega(M)$ representation

$(\hat{\mathfrak{g}} = \mathrm{Lie}(G^*))$

$\hat{\mathfrak{g}}$ の commutation relation

$$\begin{cases} [\iota_{v_i}, \iota_{v_j}] = 0 \qquad [L_{v_i}, L_{v_j}] = c_{ij}^{k} L_{v_k} \\ [L_{v_i}, \iota_{v_j}] = c_{ij}^{k} \iota_{v_k} \\ [d, \iota_{v_i}] = L_{v_i} \\ [d, L_{v_i}] = [d, d] = 0 \end{cases}$$

- equivariant cohomology

$$H_G(M) := H((M \times E)/G)$$

$(\rho, \Omega(M))$, (ρ', A) representations of G^*

assumption for A

(i) acyclicity ($\Leftrightarrow$ E : contractible)

(ii) condition (C) ($\Leftrightarrow$ $G \curvearrowright E$: locally free)

$$(\Omega(M) \otimes A)_{bas} := \{ G\text{-inv. and } \operatorname{Exp}(\iota_{v_i})(\omega) = \omega \}$$

($G \subset G^*$ and $\mathrm{Lie}(G) \cong \mathfrak{g}_0$)

$$H_G(M) \cong H((\Omega(M) \otimes A)_{bas})$$

- V. W. Guillemin, S. Sternberg
  Supersymmetry and equivariant de Rham theory
  より引用

§ Scheme and Superscheme

⊙ Scheme

k: 可換環　R: 可換 k-alg.

- affine space

$$\mathbb{A}^m : \{k\text{-alg.}\} \longrightarrow \{\text{sets}\}$$

$$\mathbb{A}^m(A) := \underbrace{A \times \cdots \times A}_{m\text{個}}$$

$$\mathbb{A}^m(\varphi)(a_1, \cdots, a_m) := (\varphi(a_1), \cdots, \varphi(a_m))$$

$$(\varphi \in \mathrm{Mor}(A,B) \quad a_1, \cdots, a_m \in A)$$

- affine scheme

$$\mathrm{Sp}_k R : \{k\text{-alg.s}\} \longrightarrow \{\text{sets}\}$$

$$\mathrm{Sp}_k R(A) := \mathrm{Hom}_k(R, A)$$

$$\mathrm{Sp}_k R(\varphi)(\alpha) := \varphi \circ \alpha$$

$$(\alpha \in \mathrm{Sp}_k R(A) \quad \varphi \in \mathrm{Mor}(A,B))$$

- $I \subset R$: ideal
- Open subfunctor $V(I) \subset \mathrm{Sp}_k R$

$$V(I)(A) := \{\alpha \in \mathrm{Hom}(R,A)\,;\, \alpha(I) \not\subset \mathfrak{m}$$

$\forall \mathfrak{m}$: 極大イデアル}

$$\begin{cases}\alpha_4(e_1)=0\\ \alpha_4(e_2)=-2\end{cases}\quad \cdots\cdots\ \{e_4\}$$

$$\begin{cases}\alpha_5(e_1)=i\\ \alpha_5(e_2)=-1\end{cases}\quad \cdots\cdots\ \{f_1+if_2\}$$

$$\begin{cases}\alpha_6(e_1)=-i\\ \alpha_6(e_2)=-1\end{cases}\quad \cdots\cdots\ \{f_1-if_3\}$$

simple

negative

- Killing form

$\mathrm{str}(\mathrm{ad}^2)$

$$4e_1^2+2e_3e_4+4e_4^2$$ 退化形式

cancel

$$\mathrm{str}\begin{pmatrix}X_{11} & X_{12}\\ X_{21} & X_{22}\end{pmatrix} = \cancel{\mathrm{tr}(X_{11})\ldots}$$
$$=\mathrm{tr}(X_{11})-\mathrm{tr}(X_{22}$$

- maximal torus

$$\left(\begin{array}{cc|cc}\sqrt{1-a^2} & a & 0 & 0\\ -a & \sqrt{1-a^2} & 0 & 0\\ \hline 0 & 0 & b & 0\\ 0 & 0 & 0 & b^{-1}\end{array}\right)$$

- Borel subsupergroup

$$\left(\begin{array}{cc|cc}\sqrt{1-a^2} & a & 0 & \alpha+B\\ -a & \sqrt{1-a^2} & 0 & -i(\alpha-B)\\ \hline b(-\sqrt{1-a^2}-ia)\alpha+b(-\sqrt{1-a^2}+ia)B & b(-a+i\sqrt{1-a^2})\alpha+b(-a-i\sqrt{1-a^2})B & b & c\\ 0 & 0 & 0 & b^{-1}\end{array}\right)$$

§ OSp(2|2;ℂ)

$$
\left(\begin{array}{cc|cc}
a & b & \alpha & \beta \\
c & d & \gamma & \delta \\
\hline
\varepsilon & \phi & e & f \\
\lambda & \mu & g & h
\end{array}\right)
\qquad
\begin{cases}
a^2+c^2+2\varepsilon\lambda=1 \\
b^2+d^2+2\phi\mu=1 \\
eh-fg-\alpha\beta+\gamma\delta=1 \\
ab+cd+\phi\lambda+\varepsilon\mu=0 \\
a\alpha+c\gamma+g\varepsilon-e\lambda=0 \\
a\beta+c\delta+h\varepsilon-f\lambda=0 \\
b\alpha+d\gamma+\cancel{h\varepsilon}+g\phi-e\mu=0 \\
b\beta+d\delta+h\phi-f\mu=0
\end{cases}
$$

- Lie superalgebra

$$
osp(2|2)=\left\{\left(\begin{array}{cc|cc}
0 & a & \alpha & \beta \\
-a & 0 & \gamma & \delta \\
\hline
-\beta & -\delta & b & c \\
\alpha & \gamma & d & -b
\end{array}\right)\right\}
$$

- ルート分解

$\{e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4\}$

Cartan subalg. $\{e_1, e_2\}$

$\begin{cases}\alpha_1(e_1)=0 \\ \alpha_1(e_2)=2\end{cases}$ ... $\{e_3\}$

$\begin{cases}\alpha_2(e_1)=i \\ \alpha_2(e_2)=1\end{cases}$ ... $\{f_2+if_4\}$

$\begin{cases}\alpha_3(e_1)=-i \\ \alpha_3(e_2)=1\end{cases}$ ... $\{f_2-if_4\}$

simple

positive

- ルート分解

$$\mathfrak{g} = \underbrace{\mathfrak{g}_0}_{\text{Lie alg.}} \oplus \underbrace{\mathfrak{g}_1}_{\text{可換代数}}$$

$$\mathrm{Car}(\mathfrak{g}) := \mathrm{Car}(\mathfrak{g}_0)$$

$\{e_1\}$: Cartan subalg.

$\alpha_1 = 2 \quad \cdots \quad \{e_2\}$ ~~simp~~ $\Big\}$ positive root

$\alpha_2 = 1 \quad \cdots \quad \{f_2\}$ simple

$\alpha_3 = -2 \quad \cdots \quad \{e_3\}$ $\Big\}$ negative root

$\alpha_4 = -1 \quad \cdots \quad \{f_1\}$ simple

- Borel subsupergroup

$$\left(\begin{array}{c|cc} 1 & 0 & B \\ -hB & h & 0 \\ 0 & c & h^{-1} \end{array}\right)$$

- Weyl 群

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

§ $OSp(1|2;\mathbb{C})$

$$\left(\begin{array}{c|cc} a & \alpha & \beta \\ \hline \gamma & b & c \\ \delta & d & e \end{array}\right) \qquad \begin{cases} a^2+2\gamma\delta=1 \\ be-cd-\alpha\beta=1 \\ a\alpha+d\gamma-b\delta=0 \\ a\beta+e\gamma-c\delta=0 \end{cases}$$

$$\left(\begin{array}{c|cc} 1-\gamma\delta & -d\gamma+b\delta & -\frac{cd+1}{b}\gamma+c\delta \\ \hline \gamma & b & c \\ \delta & d & \frac{cd+1}{b}+\frac{1}{b}\gamma\delta \end{array}\right) \qquad (b\neq 0)$$

Lie superalgebra

$\mathrm{Lie}(OSp(m|n;\mathbb{C}))$

$\cong \{X\in \mathrm{Mat}_{m|n\times m|n}(\mathbb{C})\ ;\ {}^{st}X\cdot T_{m|n}+T_{m|n}\cdot X=0\}$

$=: osp(m|n)$

$$osp(1|2)=\left\{\left(\begin{array}{c|cc} 0 & \alpha & \beta \\ \hline -\beta & a & b \\ \alpha & c & -a \end{array}\right)\right\}$$

$[e_1,e_2]=2e_2$ $\quad [e_1,e_3]=-2e_3$ $\quad [e_2,e_3]=e_1$

$[e_1,f_1]=-f_1$ $\quad [e_1,f_2]=f_2$ $\quad [e_2,f_1]=-f_2$

$[e_2,f_2]=0$ $\quad [e_3,f_1]=0$ $\quad [e_3,f_2]=-f_1$

$[f_1,f_1]=0$ $\quad [f_1,f_2]=-e_1$ $\quad [f_2,f_2]=0$

- $OSp(m|n)$ n: even (orthosymplectic supergr.)

$$T_{m|n} = \left(\begin{array}{c|cc} E_m & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & E_{n/2} \\ & -E_{n/2} & 0 \end{array}\right)$$

$$A \mapsto \{ X \in GL(m|n;A) ; {}^{st}X\, T_{m|n}\, X = T_{m|n} \}$$

- $Per(m)$ (periplectic supergroup)

$$T_m = \left(\begin{array}{c|c} 0 & E_m \\ \hline -E_m & 0 \end{array}\right)$$

$$A \mapsto \{ X \in GL(m|n;A) ; {}^{st}X\, T_m\, X = T_m \}$$

$$\Delta(T_{ij}) = \sum_{k=1}^{m} T_{ik} \otimes T_{kj} + \sum_{l=1}^{n} S_{i,m+l} \otimes S_{m+l,j}$$

$$\Delta(S_{ij}) = \sum_{k=1}^{m} T_{ik} \otimes S_{kj} + \sum_{l=1}^{n} S_{i,m+l} \otimes T_{m+l,j}$$

$(1 \leqq i \leqq m,\ m+1 \leqq j \leqq m+n)$

$$\Delta(S_{ij}) = \sum_{k=1}^{m} S_{ik} \otimes T_{kj} + \sum_{l=1}^{n} T_{i,m+l} \otimes S_{m+l,j}$$

$(m+1 \leqq i \leqq m+n,\ 1 \leqq j \leqq m)$

- $SL(m|n)$

  $A \longmapsto \{X \in GL(m|n;A);\ \mathrm{Ber}(X) = 1\}$

  $\mathrm{Sp}_k R \cong SL(m|n)$ s.t.

  $R = k[T_{ij}, S_{kl}] / (\mathrm{Ber} - 1)$

- Supertra~~ce~~nspose

  $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ $\quad {}^{st}X := \begin{pmatrix} {}^{t}X_{11} & {}^{t}X_{21} \\ -{}^{t}X_{12} & {}^{t}X_{22} \end{pmatrix}$

§ Example

$k = \mathbb{C}$

- $GL(m|n)$

$A \longmapsto \{X \in Mat_{m|n \times m|n}(A) \,;\, \exists X^{-1}\} =: GL(m|n;A)$

$\exists R$ s.t. $GL(m|n) \cong Sp_{\mathbb{C}} R$

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

$$X_{red} = \begin{pmatrix} (X_{11})_{red} & 0 \\ 0 & (X_{22})_{red} \end{pmatrix}$$

$$\det(X_{red}) = \det(X_{11})_{red} \det(X_{22})_{red}$$

$$R = \frac{k[T_{ij},\ 1 \leqq i,j \leqq m]}{\{(\det_0)^m \,;\, m \in \mathbb{N}\}} \otimes \frac{k[T_{ij},\ m+1 \leqq i,j \leqq m+n]}{\{(\det_1)^m \,;\, m \in \mathbb{N}\}}$$
$$\otimes\, k[S_{ij} \,;\, 1 \leqq i \leqq m,\ m+1 \leqq j \leqq m+n]$$
$$\otimes\, k[S_{ij} \,;\, m+1 \leqq m+n,\ 1 \leqq j \leqq m]$$

$$T_{ij} T_{i'j'} = T_{i'j'} T_{ij}$$

$$T_{ij} S_{kl} = S_{kl} T_{ij}$$

$$S_{ij} S_{kl} = -S_{kl} S_{ij}$$

§ Lie superalgebra

G : supergroup over $\Bbbk$

$x \in G(\Bbbk)$

$I_x := \{ f \in \Bbbk[G] ;\ f(G)(x) = 0 \}$ ideal

$\mathrm{Lie}(G) := (I_e / I_e^2)^*$

$\mu, \nu \in \Bbbk[G]^*$

$$\mu\nu : \Bbbk[G] \xrightarrow{\Delta} \Bbbk[G] \otimes \Bbbk[G] \xrightarrow{\mu \otimes \nu} \Bbbk$$

$X, Y \in \mathrm{Lie}(G)$ s.t. $X = [\mu]$ $Y = [\nu]$

$$[X, Y] := [\mu\nu] - (-1)^{\hat{x}\hat{y}} [\nu\mu]$$

(i) $[X, Y] = -(-1)^{\hat{x}\hat{y}} [Y, X]$

(ii) $[X, [Y, Z]] + (-1)^{(\hat{x}+\hat{y})\hat{z}} [Z, [X, Y]]$

$+ (-1)^{\hat{x}(\hat{y}+\hat{z})} [Y, [Z, X]] = 0$

- covering

$F,\ F_\lambda\ (\lambda \in \Lambda) : \{k\text{-alg.s}\} \longrightarrow \{\text{sets}\}$

$F_\lambda \subset F \quad \forall \lambda \in \Lambda$

$\bigcup_{\lambda \in \Lambda} F_\lambda(A^\times) = F(A^\times) \quad \forall A : k\text{-alg.}$

- scheme

$X : \{k\text{-alg.s}\} \longrightarrow \{\text{sets}\}$

(i) $X = \bigcup \{$open subfunctors of affine schemes$\}$

(ii) locality

$$\mathrm{Mor}(X,Y) \xrightarrow{\alpha} \prod_j \mathrm{Mor}(X_j, Y) \overset{\beta}{\underset{\gamma}{\rightrightarrows}} \prod_{j,j'} \mathrm{Mor}(X_j \cap X_{j'}, Y)$$

: exact

for $\forall \{X_j\}$: covering of X

$\alpha(f) := (f|_{X_j})_j$

$\beta(f_j) := (f_j|_{X_j \cap X_{j'}})_{j,j'}$

$\gamma(f_j) := (f_{j'}|_{X_{j'} \cap X_j})_{j,j'}$

$k[X] := \mathrm{Mor}(X, \mathbb{A}^1)$

- algebraic group

$G : \{k\text{-alg.s}\} \longrightarrow \{\text{groups}\}$

with scheme structure

Yoneda's lemma

$\mathrm{Mor}(\mathrm{Sp}_k R, X) \cong X(R)$

$m_G : G(A) \times G(A) \longrightarrow G(A) \quad (A : k\text{-alg.})$

$\Updownarrow$

$\Delta_G : k[G] \longrightarrow k[G] \otimes k[G]$

- superscheme

$\{k\text{-alg.s}\} \dashrightarrow \{k\text{-superalg.s}\}$