Projective embeddings and Lagrangian fibrations of singular Kummer surfaces

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§1. Introduction

$L \to X$ a polarized mfd / $\mathbb{C}$

$H^0(X, L^k)$ = holo. sections

$\exists$ natural basis for

X = toric var, abelian var, ...

Toric var :

monomials $\leftrightarrow$ lattice points in the moment polytope.

$\parallel$

image of the moment map

$T^n \curvearrowright X^n \longrightarrow \Delta \subset \mathbb{R}^n$

Abelian var :

theta fns $\vartheta\begin{bmatrix} 0 \\ * \end{bmatrix}$ (or $\vartheta\begin{bmatrix} * \\ 0 \end{bmatrix}$)

$*$ = lattice pt in T^n

Such basis are determined by Lagrangian fibration $\quad (\omega \in c_1(L))$

$\pi : X \longrightarrow \Delta$

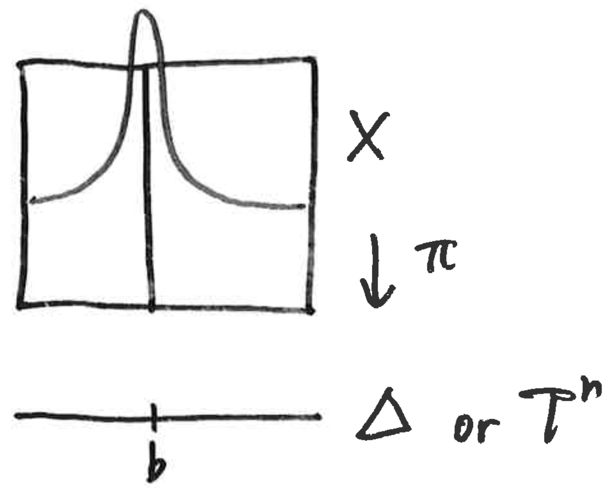
$\pi : X \longrightarrow T^n$

Lattice pts $b \in \Delta$ or T^n are characterized by

$(L^k|_{\pi^{-1}(b)}, \nabla)$: trivial

Bohr-Sommerfeld condition
(k- BS)

$\left.\begin{array}{l}\text{monomials}\\ \text{theta fns}\end{array}\right\}$ have peaks along k-BS fibers



$\dim H^0(X, L^k)$

$= \#$ of k-BS fibers

for $X =$ toric var.

abelian var.

flag var.

K3 surf.

moduli sp. of v. b.
over Riem. surf.

⋮

Ⅲ Mirror symmetry for abelian var.
(Polishchuk-Zaslow, Fukaya)

$$\begin{array}{ccc} X & & \check{X} \\ \pi \downarrow & & \check{\pi} \downarrow \\ T^n & & T^n \end{array}$$

dual torus fibration

$$L^k \to X \longleftrightarrow S_k \subset \check{X}$$

Lagranglan section

$$\mathcal{O}_X = L^0 \longleftrightarrow S_0 \subset \check{X}$$

zero section

$$H^0(X, L^k) \; (= \mathrm{Hom}(L^0, L^k)) \cong HF(S_0, S_k)$$

Floer homology

$$\cong \bigoplus_{b \in S_0 \cap S_k} \mathbb{C}\, b$$

$$\vartheta\begin{bmatrix} 0 \\ b \end{bmatrix} \longleftrightarrow b$$

<u>Rem.</u> $b \in S_0 \cap S_k \; (\subset T^n) \iff \pi^{-1}(b)$: k-BS.

(III) Proj. embeddings and Lag. fibrations

$s_0, \cdots, s_{N_k}$ basis of $H^0(X, L^k)$

$$\begin{array}{cccc} \iota_k : & X & \hookrightarrow & \mathbb{C}P^{N_k} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & z & \longmapsto & (s_0(z) : \cdots : s_{N_k}(z)) \end{array}$$

- Std. Lag. fibration of $\mathbb{C}P^{N_k}$
  = moment map of T^{N_k}-action

$$\begin{array}{cccc} \mu_k : & \mathbb{C}P^{N_k} & \longrightarrow & \Delta_k \subset (\mathrm{Lie}\, T^{N_k})^* \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & (Z_0 : \cdots : Z_{N_k}) & \longmapsto & \frac{1}{\sum |Z_i|^2}(|Z_0|^2, \cdots, |Z_{N_k}|^2) \end{array}$$

Restrict to X :

$$\pi_k := \mu_k \circ \iota_k : X \longrightarrow B_k \subset \Delta_k$$

Compare $\begin{cases} \pi : X \to B \; (= \Delta, T^n, \cdots) \\ \pi_k : X \to B_k \end{cases}$

Toric var:

$s_0, \cdots, s_{N_k}$ = monomials.

$\Rightarrow \iota_k : X^n \longrightarrow \mathbb{C}P^{N_k}$: T^n-equivariant.

$\Rightarrow \pi = \pi_k$

Abelian var:

$s_i = \vartheta\begin{bmatrix} 0 \\ * \end{bmatrix}$ theta fn.

$\pi_k \neq \pi$

Thm (N.)

π_k converges to π in a suitable sense ("Gromov-Hausdorff convergence) as $k \to \infty$.

Today: X = singular Kummer surf.

§2. Theta functions and projective embeddings

$A = \mathbb{C}^2/\Omega\mathbb{Z}^2 + \mathbb{Z}^2$: abelian surf.

$L \to A$: ample, symm, $\deg = 1$.

$\left((-1)_A^* L \cong L \right)$

$$\left(\begin{array}{l} L = (\mathbb{C}^2 \times \mathbb{C})/(\Omega\mathbb{Z}^2 + \mathbb{Z}^2), \\ (z, \xi) \sim \left(z+\lambda, \exp\left(\pi {}^t\bar{\lambda}(\mathrm{Im}\Omega)^{-1} z + \frac{\pi}{2} {}^t\bar{\lambda}(\mathrm{Im}\Omega)^{-1}\lambda\right)\xi\right) \end{array} \right)$$

$T^f = T^b = \mathbb{R}^2/\mathbb{Z}^2$

$A \cong T^f \times T^b$, $\quad z = \Omega x + y \leftrightarrow (x, y)$.

$\omega_0 = \frac{\sqrt{-1}}{2} {}^t dz (\mathrm{Im}\Omega)^{-1} d\bar{z} = -\sum_\alpha dx^\alpha \wedge dy^\alpha$

: flat metric in $c_1(L)$.

$\pi : (A, \omega_0) \longrightarrow T^b$

natural proj. : Lag. fibration.

For $k \in \mathbb{N}$,

$$T_k^{f(b)} = \frac{1}{k}\mathbb{Z}^2/\mathbb{Z}^2 \subset T^{f(b)}$$

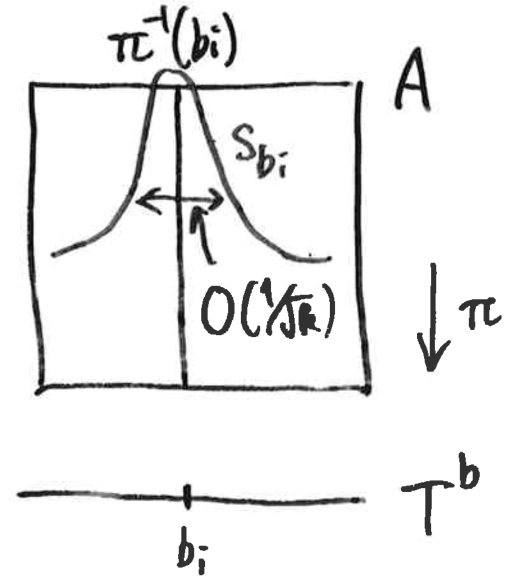
$$A_k = T_k^f \times T_k^b$$

$$T_k^b = \{b_i\}_{i=1,\cdots,k^2}$$

$$\Rightarrow \quad s_{b_i} = s_i = C_k \exp\left(\frac{k\pi}{2} {}^t z (\mathrm{Im}\,\Omega)^{-1} z\right) \vartheta\begin{bmatrix}0\\-b_i\end{bmatrix}\left(\frac{\Omega}{k}, z\right)$$

: ONB of $H^0(A, L^k)$.

$$\begin{cases} \vartheta\begin{bmatrix}a\\b\end{bmatrix}(\Omega, z) = \sum_{n\in\mathbb{Z}^2} e\Big(\frac{1}{2}{}^t(n+a)\Omega(n+a) \\ \qquad\qquad\qquad\qquad\qquad + {}^t(n+a)(z+b)\Big) \\ e(t) = \exp(2\pi\sqrt{-1}\,t). \end{cases}$$



$$\begin{array}{rccc} \iota_k : & A & \hookrightarrow & \mathbb{C}P^{k^2-1} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & z & \longmapsto & (s_{b_1}(z) : \cdots : s_{b_{k^n}}(z)) \end{array}$$

$$\omega_k = \frac{1}{k} \iota_k^* \omega_{FS} \in G(L)$$

$$\pi_k = \mu_k \circ \iota_k : A \longrightarrow B_k \subset \Delta_k$$

<u>Thm</u> (N.)

(1) $\omega_k \longrightarrow \omega_0$ in C^∞

In particular,

$(A, \omega_k) \longrightarrow (A, \omega_0)$ Gromov-Hausdorff

(2) $B_k \longrightarrow T^b$: G-H

(3) $\{\pi_k : (A, \omega_k) \to B_k\}_k$ converges to $\pi : (A, \omega_0) \to T^b$ as a map between metric spaces.

(iii) distance on B_k.

Define a metric on Δ_k so that

$$\mu_k : (\mathbb{CP}^{N_k}, \tfrac{1}{k}\omega_{FS}) \longrightarrow \Delta_k$$

is Riemannian submersion.

Equivalently :

$$(\mathbb{RP}^{N_k}, \tfrac{1}{k}\omega_{FS}) \longrightarrow \Delta_k$$

branched covering

is locally isometric.

$\rightsquigarrow$ induced distance on $B_k \subset \Delta_k$.

Idea of Proof

(1)

Thm (Tian, Zelditch)

(L, h) : Hermitian line bdle

$\downarrow$

(X, ω) : cpt Kähler $\qquad \omega = c_1(L, h)$

$s_0^k, \cdots, s_{N_k}^k \in H^0(X, L^k)$ ONB.

$$\Rightarrow \quad \|\omega - \omega_k\|_{C^r} = O(1/k)$$

(2)

$\left.\begin{array}{l} \pi : (A, \omega) \longrightarrow T^b \\ \pi_k : (A, \omega_k) \longrightarrow B_k \end{array}\right\}$ invariant under T^f_k-action (translation)

$$\Rightarrow \quad \begin{array}{ccc} (X/T^f_k, \omega) & \qquad & (X/T^f_k, \omega_k) \\ \downarrow & & \downarrow \\ T^b & & B_k \end{array}$$

diam. of fibers $= O(\frac{1}{k})$

§3. Line bundles on Kummer surfaces.

$$\begin{array}{cccc} (-1)_A : & A & \longrightarrow & A \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & z & \longmapsto & -z \end{array}$$: inverse morphism.

$e_1, \cdots, e_{16} \in A_2$: half-periods.
i.e. $(-1)_A e_i = e_i$

$$\begin{tikzcd} Z_i \arrow[rr, bend left] & & e_i \\ \tilde{A} \arrow[u, phantom, "\cup" sloped] \arrow[rr, "q"] \arrow[d, "\tilde{p}"'] & & A \arrow[u, phantom, "\ni" sloped] \arrow[d, "p"] \\ \tilde{X} \arrow[rr] & & X = A/(-1)_A \end{tikzcd}$$

Kummer surface — singular Kummer surface.

$L \to A$: symmetric line bdl. : $(-1)_A^* L \cong L$

$$\begin{tikzcd} L \arrow[r, "(-1)_L"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & L \arrow[d] \\ A \arrow[r, "(-1)_A"'] & A \end{tikzcd}$$, $(-1)_L^2 = 1$.

$\Rightarrow$

$(-1)_L = \pm 1$ on L_{e_i}

e_i : even (resp. odd)

$:\Leftrightarrow (-1)_L = 1$ (resp. -1) on L_{e_i}

$n^{\pm}(L) :=$ # of even/odd half periods.

$$Z^{\pm} := \sum_{e_i : \substack{\text{even} \\ \text{odd}}} Z_i \subset \tilde{A}$$

$$\begin{array}{ccc} H^0(A, L) & \longrightarrow & H^0(A, L) \\ \Psi & & \Psi \\ s & \longmapsto & (-1)_L \, s \, (-1)_A \end{array}$$

$$\begin{tikzcd} L \arrow[r, "(-1)_L"] & L \\ A \arrow[u, "s"] \arrow[r, "(-1)_A"'] & A \arrow[u, "(-1)_L s (-1)_A"'] \end{tikzcd}$$

: involution.

$H^0(A, L)^{\pm} := \pm 1$ - eigenspace

even / odd sections.

<u>Prop</u> $\tilde{p}_* q^* L$: loc. free, $rk = 2$.

$$\tilde{p}_* q^* L = M^+ \oplus M^-.$$

$$H^0(\tilde{X}, M^{\pm}) \cong H^0(A, L)^{\pm}.$$

$$\begin{tikzcd} \tilde{A} \arrow[r, "q"] \arrow[d, "\tilde{p}"'] & A \arrow[d, "p"] \\ \tilde{X} \arrow[r] & X \end{tikzcd}$$

<u>Prop</u> $\tilde{p}^* M^{\pm} = q^* L \otimes \mathcal{O}_{\tilde{A}}(-Z^{\mp})$

<u>Thm</u> (T. Bauer) L : ample, symm.

$$\dim H^0(M^{\pm}) = 2 + \frac{1}{2} \dim H^0(L) - \frac{1}{4} n^{\mp}(L)$$

$$(= \chi(M^{\pm}))$$

⑪ Projective images.

$L \to A$: ample symm. deg = 1.

$M_k^{\pm} \to \tilde{X}$; corresponding to $L^k \to A$

<u>Thm</u> (Bauer)

$k \geq 4$ for M_k^+

$k > 4$ for M_k^-.

- $M_k^{\pm}$: base point free.
- $\hat{X} \longrightarrow \mathbb{P}(H^0(M_k^{\pm})^*)$

  birat. onto its image.

  contracted curves = D_i with $e_i \begin{cases} \text{even} \\ \text{odd} \end{cases}$

Note : $\forall e_i$ even for L^{2k}.

<u>Cor</u> $X \hookrightarrow \mathbb{P}(H^0(M_{2k}^+)^*)$, $k \geq 2$.

<u>Prop</u>. $M_{2k}^+ = (M_2^+)^k$

§4. Main Theorem

$L \to A$ as in §2.

$M = M_2^+ \longrightarrow X$

$$\begin{array}{ccc} H^0(A, L^k) & \longrightarrow & H^0(A, L^k) \\ \Psi & & \Psi \\ s & \longmapsto & (-1)_{L^k}\, s\, (-1)_A \end{array}$$

$$(-1)_{L^k}\, s_{b_i}\, (-1)_A = s_{-b_i}$$

$\Rightarrow$ $H^0(A, L^k)^\pm$ generated by $s_{b_i} \pm s_{-b_i}$

<u>Rem</u>

$$\dim H^0(L^{2k})^+ = 2k^2 + 2 = \frac{(2k)^2 - 4}{2} + 4$$

$$\dim H^0(L^{2k})^- = 2k^2 - 2 = \frac{(2k)^2 - 4}{2}$$

ω : flat orbifold metric in $c_1(M)$.

$$\begin{array}{ccc} \pi : X & \longrightarrow & B \approx S^2 \\ \| & & \| \\ A/(-1) & & T^b/(-1) \end{array}$$

$\uparrow$

w/ flat orbifold metric

$$t_i = \begin{cases} \frac{1}{2}(s_{b_i} + s_{-b_i}) , & b_i \in T^b_{2k} \setminus T^b_2 \\ \frac{1}{\sqrt{2}} s_{b_i} , & b_i \in T^b_2 \end{cases}$$

: ONB of $H^0(X, M^k)$.

$$\begin{array}{cccc} \iota_k : & X & \hookrightarrow & \mathbb{C}P^{N_k} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & z & \longmapsto & (t_0(z) : \cdots \quad) \end{array}$$

$$\omega_k = \frac{1}{k} \iota_k^* \omega_{FS}$$

$$\pi_k = \mu_k \circ \iota_k : X \longrightarrow B_k \subset \Delta_k$$

<u>Thm</u>

Assume : Ω is pure imaginary

Then

(1) $(X, \omega_k) \longrightarrow (X, \omega)$ Gromov-Hausdorff

(2) $B_k \longrightarrow B$: G-H

(3) $\pi_k \longrightarrow \pi$

<u>Rem</u> on the assumption for Ω

- not necessary for (1)
- might not be crucial ?

Proof of (1)

Thm (J. Song)

(X^n, ω) : cpt Kähler orbifold, $n \geq 2$

(with isolated sing. $\{z_j\}_{j=1}^m$)

$(M, h) \to X$: orbifold line bdl.

with $c_1(M, h) = \omega$

$t_0^k, \cdots, t_{N_k}^k \in H^0(X, M^k)$: ONB

$$\Rightarrow \left| \sum_{i=0}^{N_k} |t_i^k(z)|_h^2 - \sum_{l=0}^{R-1} a_l(z) k^{n-l} \right|_{C^q, z}$$

$$\leq C_{R,q} \left(k^{n-R} + k^{n+\frac{q}{2}} e^{-\delta k r(z)^2} \right)$$

where

$r(z) = \min_j \{ \mathrm{dist}(z, z_j) \}$

$a_0 = 1$

$$\left(\begin{array}{l} a_1 = \text{scalar curvature of } (X, \omega) \\ \vdots \end{array} \right)$$

$| \cdot |_{C^q, z}$ = C^q-norm at z

Cor

$(M, h) \to (X, \omega)$ as above.

$$\begin{array}{rccc} \iota_k : & X & \hookrightarrow & \mathbb{C}P^{N_k} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & z & \longmapsto & (t_0^k(z) : \cdots : t_{N_k}^k(z)) \end{array}$$

$$\omega_k := \frac{1}{k} \iota_k^* \omega_{FS}$$

$$\Rightarrow \quad |\omega_k - \omega|_{C^q, z} \leq C_q \left(\frac{1}{k} + k^{\frac{q}{2}} e^{-\delta k r(z)^2} \right)$$

In particular,

$\omega_k \to \omega$ in C^∞

on cpt sets $\subset X \setminus \{z_j\}_{j=1}^{m}$

<u>Prop</u> $z \in X$.

$d(z, z_j)$ = distance w.r.t. ω

$d_k(z, z_j)$ = distance w.r.t. ω_k

$$\left| d(z, z_j) - d_k(z, z_j) \right| \leq O(1/\sqrt{k})$$

$$\left(\int_0^1 e^{-\delta k t^2} dt = O(1/\sqrt{k}). \right)$$

Proof of (2)

Construct $\varphi_k : B \longrightarrow B_k$ as follows.

$0 \in T_2^f$

$B \simeq (\{0\} \times T^b)/(-1) \subset X$ "zero section"

$\varphi_k = \pi_k|_B : B \longrightarrow B_k$

Note: $B \hookrightarrow X$ is an isometric embedding.

Prop

$\forall \varepsilon > 0$, $\exists k_0 (= O(1/\varepsilon^2))$ s.t.

$\varphi_k : B \to B_k$ is an ε-Hausdorff approximation for $k \geq k_0$

(iii)

$$\begin{tikzcd} M \arrow[r, "\tilde{\imath}"] \arrow[d] & M \arrow[d] \\ X \arrow[r, "\imath"] & X \\ z \arrow[u, phantom, "\in" sloped] \arrow[r, mapsto] & \bar{z} \arrow[u, phantom, "\in" sloped] \end{tikzcd}$$

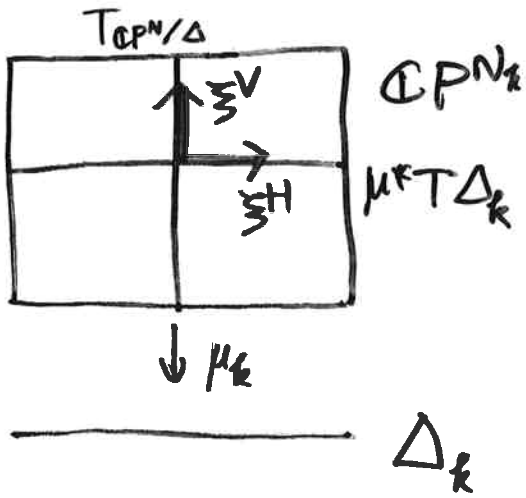
anti holo. involution

- $B \subset X$ fixed pts under $\imath$
- $\tilde{\imath}(t_i) = \bar{t}_i$

$$\begin{tikzcd} z \arrow[d, mapsto] & \in X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{C}P^{N_k} \arrow[d] & \ni (z_0 : \cdots : z_{N_k}) \arrow[d, mapsto] \\ \bar{z} & \in X \arrow[r] & \mathbb{C}P^{N_k} & \ni (\bar{z}_0 : \cdots : \bar{z}_{N_k}) \end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd} X \arrow[r, hook] & \mathbb{C}P^{N_k} \\ B \arrow[u, phantom, "\cup" sloped] \arrow[r, hook] \arrow[dr, "\varphi_k"'] & \mathbb{R}P^{N_k} \arrow[u, phantom, "\cup" sloped] \arrow[d, "\text{covering}"] \\ & \Delta_k \end{tikzcd}$$

$$\begin{array}{ccccc} T_p\mathbb{CP}^{N_k} & = & T_{\mathbb{CP}^{N_k}/\Delta_k} & \overset{\perp}{\oplus} & \mu_k^* T_{\mu_k(p)}\Delta_k \\ \cup & & \cup & & \cup \\ \xi & = & \xi^V & + & \xi^H \end{array}$$



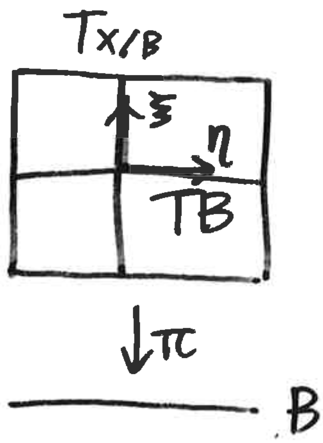
Lem Away from singularities:

i) $\xi \in T_{X/B} \subset TX$

$$|d\iota_k(\xi)^H| \leq C\varepsilon|\xi|$$

ii) $\eta \in \pi^*TB \subset TX$

$$|d\iota_k(\eta)^V| \leq C\varepsilon|\eta|$$



$$TX = T_{X/B} \overset{\perp}{\oplus} \pi^* TB$$

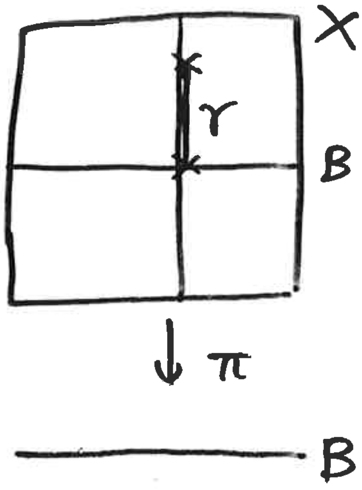
Lemma follows from

$$\frac{Z_i}{Z_0} = \frac{t_i(z)}{t_0(z)} = \exp\left(f(y) + \sqrt{-1}\, g(x) \right) + O(\varepsilon)$$

f, g : linear fns.

x : fiber

y : base



γ : a curve along a fiber of $\pi : X \to B$

$\Rightarrow$ (length of γ) $\leq O(\varepsilon)$