Poisson groupoidのtwistとゲージ変換

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目次

１. 導入

２. Poisson groupoids（復習）

３. twisting

４. 主定理

1. 導入

1.0. Poisson manifolds

(M, π), $\pi \in \Gamma \bigwedge^2 TM$: Poisson manifold
$\Longleftrightarrow$
$[\pi, \pi] = 0$ (Schouten bracket)
$\Longleftrightarrow$
$\{f, g\} := \pi(df, dg)$, $f, g \in C^\infty(M)$:
Poisson bracket, i.e.,

(P1) $\{\cdot, \cdot\}$: Lie bracket,
(P2) $\{f, gh\} = g\{f, h\} + \{f, h\}g$, $f, g, h \in C^\infty(M)$.

因みに,

$$\frac{1}{2}[\pi, \pi](df, dg, dh) = \{\{f, g\}, h\} + c.p.$$

1.1. ゲージ変換 (A.Weinstein and P.Severa)

(M, π): Poisson manifold.

B: closed 2-form on M.

$$L_\pi := \{\tilde{\pi}(a) + a | a \in T^*M\} \subset TM \oplus T^*M,$$

where $\tilde{\pi}(a) := \pi(a, \cdot)$, $a \in T^*M$.

$\tau_B : TM \oplus T^*M \to TM \oplus T^*M$, s.t.,

$$\tau_B(x + a) := x + a + \widetilde{B}(x).$$

$\tau_B^{-1} = \tau_{-B}$.

定義 1.1.

Poisson structure ${}^\exists\pi'$ on M: $L_{\pi'} = \tau_B(L_\pi)$

$\Longleftrightarrow$

$\pi \mapsto \pi'$: "gauge transformation by B",

$\pi \equiv \pi'$: "gauge equivalence".

このとき,

$\tilde{\pi}' = \tilde{\pi}(1 + \widetilde{B}\tilde{\pi})^{-1}$, where $1 + \widetilde{B}\tilde{\pi} : T^*M \to T^*M$.

補題 1.2.

π が B でgauge transformation できる

$\Longleftrightarrow$

$1+\widetilde{B}\tilde{\pi}$: invertible.

例 1.3. π_1, π_2: 非退化-Poisson structures on M.

$\widetilde{B}_{12} := \tilde{\pi}_1^{-1} - \tilde{\pi}_2^{-1}$,

$1+\widetilde{B}_{12}\tilde{\pi}_2 = \tilde{\pi}_1^{-1}\tilde{\pi}_2$.

故に, $\tilde{\pi}_1 = \tilde{\pi}_2(1+\widetilde{B}_{12}\tilde{\pi}_2)^{-1}$.

1.2. Symplectic groupoid のゲージ変換

H. Bursztyn and O. Radko の研究.

$$\begin{array}{ccc} \text{symplectic g.p.} & \xrightarrow{?} & \text{symplectic g.p.} \\ \text{積分}\Big\uparrow & & \text{積分}\Big\uparrow \\ \text{Poisson m.f.d.} & \xrightarrow{\text{by } B} & \text{Poisson m.f.d.} \end{array}$$

一般に,

ϕ: $^{\exists}$ symplectic m.f.d. $\to$ $^{\forall}$ Poisson m.f.d.,

where ϕ: Poisson map, i.e.,

ϕ^*: Poisson algebra homomorphism:

$$\phi^*\{f, g\} = \{\phi^* f, \phi^* g\}, \quad f, g \in C^\infty.$$

$\mathbf{s}, \mathbf{t} : G \rightrightarrows G_0$: symplectic groupoid,

(G, ω): symplectic manifold,

(G_0, π_0): Poisson manifold.

定理 1.4 (symplectic groupoidsのゲージ共変性.)

$\pi_0 \mapsto \pi_0'$: gauge transformation by B.

$\Rightarrow$

symplectic structureのゲージ変換:

$$\omega' = \omega + \mathbf{s}^* B - \mathbf{t}^* B.$$

結論:

1. (ある条件の下で) ゲージ同値ならば森田同値.

2. presymplectic groupoids.

(ゲージ変換が退化した場合)

1.3. Poisson groupoidsのゲージ変換.

symplectic groupoids $\subset$ Poisson groupoids.

$\mathbf{s}, \mathbf{t} : G \rightrightarrows G_0$: Poisson grouopid
$\Rightarrow$
G_0: Poisson manifold.

目的 1：Bursztyn and Radko の仕事を拡張する.

他方, ゲージ変換, それ自体の一般化が得られる:

(M, π): Poisson manifold.

$$d\Omega + \frac{1}{2}\{\Omega, \Omega\}_\pi = 0,$$

where $\Omega \in \Gamma \bigwedge^2 T^*M$.

この解Ωを用いれば, Poisson structureの変換が得られる.

$$\pi \mapsto \widetilde{P} := \tilde{\pi} + \tilde{\pi}\widetilde{\Omega}\tilde{\pi}.$$

$P \in \Gamma \bigwedge^2 TM$: Poisson structure.

目的 2：Ωによる変換の影響を調べる.

(Poisson groupoids特有)

Ωによる変換がtwist(後述)のspecial caseだと気づく.

目的 3：Poisson groupoidのtwistを行う.

ポイント:

Structures(tensors θ, s.t., $\{\theta, \theta\}_{\mathrm{big}} = 0$) の代わりにDirac structures. Big-bracketsの代わりにCourant bracketsを用いる.

つまり全部, Dirac structuresにする.

(導入終わり)

2 Poisson groupoids (復習)

2.1. Lieの定理

Lie群の類似物

$\mathbf{t}, \mathbf{s} : G \rightrightarrows G_0$: Lie groupoid (smooth invertible small category).

簡単に, $G \rightrightarrows G_0$ or Gとかく.

因みに, $G_0 = \{e\} \Rightarrow G$はLie群, eは単位元.

Lie環の類似物

$A \to M$: vector bundle,
$\sigma : A \to TM$: bundle map (anchor map).

(A, σ) or A: "Lie algebroid" on M
$\Longleftrightarrow$
(A1) $(\Gamma A, [\cdot, \cdot])$: Lie algebra on $\mathbf{R}$,
(A2) $[X, fY] = f[X, Y] + \sigma(X)(f)Y$, $f \in C^\infty(M)$,
(A3) $\sigma[X, Y] = [\sigma(X), \sigma(Y)]$.

Lie群のLie環の類似物

AG: "the Lie algebroid of G"

$\Longleftrightarrow$

$AG \to G_0$: vector bundle on G_0,

$\Gamma AG \cong \{\text{left invariant vector fields}\} \subset \Gamma TG$.

補題 2.2. AGはLie algebroid on G_0.

そこで, $G \rightrightarrows G_0$をAGの積分と解釈する.

例 2.3. G: Lie group $\Rightarrow$ $AG = \mathfrak{g}$.

例 2.4. $D \subset TM$: involutive subbundle on M.

$G := \bigcup(l \times l)$, where l a leaf,

$G_0 := \{(x, x) | x \in M\}$,

$\mathbf{s}(x, y) = x$, $\mathbf{t}(x, y) = y$,

$AG = D$.

例 2.5. (M,π): Poisson manifold.

$(T^*M,\tilde{\pi})$: Lie algebroidになる by

$\{\alpha,\beta\}_\pi := \mathfrak{L}_{\tilde{\pi}(\alpha)}\beta - \mathfrak{L}_{\tilde{\pi}(\beta)}\alpha + d\pi(\beta,\alpha)$,
where $\alpha,\beta \in \Gamma T^*M$.

これをT^*M_πとかく.
T^*M_0は自明なLie algebroid, 0を省略する.

因みに, $d\{f,g\} = \{df,dg\}_\pi$, $f,g \in C^\infty(M)$.

例 2.6. $A = TM \times \mathfrak{g}$.

2.2. Lie bialgebroids and Poisson groupoids

(A, σ), (A^*, σ_*): Lie algebroids on M,
$d : \Gamma \bigwedge^k A^* \to \Gamma \bigwedge^{k+1} A^*$,
$(d_* : \Gamma \bigwedge^k A \to \Gamma \bigwedge^{k+1} A)$

(A, A^*): Lie bialgebroid on M
$\Longleftrightarrow$
$d\{\alpha, \beta\} = \{d\alpha, \beta\} + \{\alpha, d\beta\}$, $\alpha, \beta \in \Gamma A^*$,
where $\{\cdot, \cdot\}$: Lie bracket on ΓA^*
$\Longleftrightarrow$
$d_*[X, Y] = [d_*X, Y] + [X, d_*Y]$, $X, Y \in \Gamma A$,
where $[\cdot, \cdot]$: Lie bracket on ΓA.

例 2.7. (M, π): Poisson manifold.
(TM, T^*M_π): Lie bialgebroid by

$d_*X := [\pi, X]$, $X \in \Gamma TM$.

Poisson Lie群の類似物

$\mathbf{s}, \mathbf{t} : G \rightrightarrows G_0$: Lie groupoid,
(G, π_G): Poisson manifold.

(G, π_G): Poisson groupoid
$\Longleftrightarrow$
$\{(x, y, xy) |$ 積の graph$\} \subset G \times G \times G^-$: coisotropic submanifold.
$\Longleftrightarrow$
(AG, AG^*): Lie bialgebroid.

例 2.8.
symplectic groupoids ($AG \cong T^*M_\pi$),
Poisson Lie groups.

例 2.9. (M, π): Poisson manifold.

$G := M \times M$, $G_0 := M$, $\mathbf{s}(x, y) := x$, $\mathbf{t}(x, y) := y$

$$\pi_G := \pi \times -\pi.$$

$AG = TM$.

命題 2.10 (a).
(G, π_G): Poisson groupoid
$\Rightarrow$
$(G_0, {}^{\exists}\pi_0)$: Poisson manifold,
$\mathbf{s}$: Poisson map, i.e.,

$$\mathbf{s}^*\{f, g\}_0 = \{\mathbf{s}^* f, \mathbf{s}^* g\}_G, \quad f, g \in C^\infty(G_0).$$

命題 2.10 (b).
(A, A^*): Lie bialgebroid on M
$\Rightarrow$
(M, π): Poisson manifold,
where

$$\tilde{\pi} := \pm \sigma_* \cdot \sigma^*.$$

$A = AG$のとき $(M = G_0)$,

(a), (b)で定まるPoisson structureは同じ(±除く).

(復習終わり)

3. Twist

(A, A^*): Lie bialgebroid on M.
The double of (A, A^*):

$$\mathbf{E} := \{A \oplus A^*, [\![\cdot,\cdot]\!], (\cdot,\cdot), \rho\},$$

i.e., $\{\mathbf{E}, A, A^*\}$: the Manin triple.
($\mathbf{E}$: Courant algebroid.)

“twisting by H”

${}^\forall H \in \Gamma \bigwedge^2 A$:

$$\tau_H(x + a) := x + \widetilde{H}(a) + a, \quad x + a \in A \oplus A^*.$$

$\tau_H^{-1} = \tau_{-H}$.

$$\tau_{-H}(\mathbf{E}) := \{A \oplus A^*, [\![\cdot,\cdot]\!]', (\cdot,\cdot)', \rho'\},$$

where

$$\begin{aligned} \tau_{-H}[\![\mathbf{x},\mathbf{y}]\!] &= [\![\tau_{-H}(\mathbf{x}), \tau_{-H}(\mathbf{y})]\!]', \\ (\mathbf{x},\mathbf{y}) &= (\tau_{-H}(\mathbf{x}), \tau_{-H}(\mathbf{y}))', \\ \rho \cdot \tau_H &= \rho'. \end{aligned}$$

つまり, $\tau_{-H}(\mathbf{E}) \cong \mathbf{E}$ (doubleとして).

定義 3.1.

$$\{\tau_{-H}(\mathbf{E}), A, A^*\}$$

を $\{\mathbf{E}, A, A^*\}$ の “twisting by H” とよぶ.

注意 3.1.1.

$$\tau_{-H}(\mathbf{E})|_{A^*} =: A^* \neq A^* := \mathbf{E}|_{A^*}.$$

命題 3.2.

$\{\tau_{-H}(\mathbf{E}), A, A^*\}$: Manin triple

$\Longleftrightarrow$

L_H(:= the graph of $\widetilde{H}$): (integrable) Dirac structure of $\mathbf{E}$

$\Longleftrightarrow$

$$d_* H + \frac{1}{2}[H, H] = 0.$$

証明.

$$\begin{aligned} \tau_H\{\tau_{-H}(\mathbf{E}), A, A^*\} &= \{\mathbf{E}, \tau_H(A), \tau_H(A^*)\} \\ &= \{\mathbf{E}, A, L_H\}. \end{aligned}$$

(解説) (A, A^*): Lie bialgebroid, $\mathbf{E}$: the double.

Dirac structures	structures
$(A, [\cdot,\cdot], \sigma)$	(A, μ)
$(A^*, \{\cdot,\cdot\}, \sigma_*)$	(A^*, η)
$\mathbf{E}$ on $A \oplus A^*$	$(A \oplus A^*, \mu + \eta)$

$$(A^*, \{\cdot,\cdot\}', \sigma'_*) := \tau_{-H}(\mathbf{E})|_{A^*}$$

$$\{\alpha, \beta\}' = \mathfrak{L}_{\widetilde{H}\alpha}\beta - \mathfrak{L}_{\widetilde{H}\beta}\alpha + dH(\beta, \alpha) + \{\alpha, \beta\},$$

where $\alpha, \beta \in \Gamma A^*$,

$$\sigma'_*(a) = -\sigma\widetilde{H}(a) + \sigma_*(a),$$

where $a \in A^*$.

$$\boxed{\mathfrak{L}_{\widetilde{H}\alpha}\beta - \mathfrak{L}_{\widetilde{H}\beta}\alpha + dH(\beta, \alpha),\ -\sigma\widetilde{H}(a) \iff \eta_{\mu,H}}$$

"Lie bialgebroidのtwist" (Kosmann-Schwarzbach, Roytenberg):

$$\mu' := \mu,$$
$$\eta' := \eta + \eta_{\mu,H}, \quad \text{3-form} := d_*H + \frac{1}{2}[H, H].$$

例/補題 3.3.

(M, π): Poisson manifold.

(TM, T^*M_π): Lie bialgebroid.

$\mathbf{E}_\pi$: the double of (TM, T^*M_π).

$$\tau_\pi(\mathbf{E}_\pi) = \mathbf{E}_0, \quad (\pi = 0).$$

故に,

$\{\mathbf{E}_\pi, TM, T^*M_\pi\}$: Manin triple

$\Longleftrightarrow$

$\{\mathbf{E}_0, TM, L_\pi\}$: Manin triple

$\Longleftrightarrow$

$[\pi, \pi] = 0$, where $d_* = 0$.

補題 3.4. (M, π): Poisson manifold with closed 2-form B.

πがBでゲージ変換できる

$\Longleftrightarrow$

$L_{-B} \cap L_\pi = 0$

$\Longleftrightarrow$

$\{\mathbf{E}_0, L_{-B}, L_\pi\}$: Manin triple.

因みに, このとき(L_{-B}, L_π)はLie bialgebroidで, 導かれるPoisson structureがゲージ変換: $\pi \mapsto \pi'$.

ゲージ変換の一般化

$\{\mathbf{E}_0, L, L_\pi\}$: Manin triple

$\Longleftrightarrow$

L: (integrable) Dirac structure, s.t., $L \cap L_\pi = 0$

$\Longleftrightarrow$

(L, L_π): Lie bialgebroid.

$\tau_\pi(\mathbf{E}_\pi) = \mathbf{E}_0$ から次を得る.

$L \cap L_\pi = 0 \iff \tau_{-\pi}(L) = L_\Omega$, $({}^\exists\Omega \in \Gamma \bigwedge^2 T^*M)$.

故に,

L: (integrable) Dirac structure of $\mathbf{E}_0$

$\Longleftrightarrow$

L_Ω: (integrable) Dirac structure of $\mathbf{E}_\pi$

$\Longleftrightarrow$

$$d\Omega + \frac{1}{2}\{\Omega, \Omega\}_\pi = 0.$$

例 3.5. $L_{-B} \cap L_\pi = 0$:

$\tau_{-\pi}(L_{-B}) = L_{\Omega_B}$, where $\widetilde{\Omega}_B = -\tilde{B}(1+\tilde{\pi}\tilde{B})^{-1}$.

$\tilde{P} := \tilde{\pi} + \tilde{\pi}\widetilde{\Omega}_B\tilde{\pi}$ とおく.

これが B によるゲージ変換: $\pi \mapsto P = \pi'$ になる.

例 3.6. $L_{-\pi_1} \cap L_\pi = 0$:

$\tau_{-\pi}(L_{-\pi_1}) = L_\Omega$, where $\widetilde{\Omega} = -(\tilde{\pi}_1 + \tilde{\pi})^{-1}$.

応用 3.7. (M, π_s): symplectic manifold with Poisson structure π.

$$\tilde{\pi} = \tilde{\pi}_s + \tilde{\pi}_s\widetilde{\Omega}\tilde{\pi}_s. \quad ({}^\exists\Omega.)$$

4. 主定理

上述のMaurer-Cartan方程式の解を "Hamilton operator" とよぶ.

<u>定理 A.</u>

(G, π_G): Poisson grouopid on (G_0, π_0),

Ω: Hamilton operator on (G_0, π_0)

$\Rightarrow$

$\mathbf{s}^*\Omega - \mathbf{t}^*\Omega$: Hamilton operator on (G, π_G),

(G, P_G): Poisson groupoid on (G_0, P_0), where

$$\begin{aligned} P_G &:= \pi_G + \pi_G(\mathbf{s}^*\Omega - \mathbf{t}^*\Omega)\pi_G, \\ P_0 &:= \pi_0 + \pi_0\Omega\pi_0 \end{aligned}$$

$\Omega = \Omega_B$ のとき,

$$\mathbf{s}^*\Omega_B - \mathbf{t}^*\Omega_B = \Omega_{\mathbf{s}^*B - \mathbf{t}^*B}.$$

<u>系 A-1.</u> Poisson grouopid はゲージ共変的.

$d\Omega = \{\Omega, \Omega\}_\pi = 0$ のとき,

$t\Omega$: Hamilton operator for each $t \in \mathbf{R}$.

系 A-2. このとき，Poisson grouopid の変形が得られる.

ここで，Lie bialgebroid (AG, AG^*) を思い出す.

$\sigma_* : AG^* \to TM$: anchor map.

$H := \sigma_*^* \cdot \Omega \cdot \sigma_* \in \Gamma \bigwedge^2 AG$.

$$d_* H + \frac{1}{2}[H, H] = 0,$$

i.e., H: Hamilton operator.

かつ,

$$\begin{aligned} \overleftarrow{H} &= \pi_G \cdot \mathbf{s}^*\Omega \cdot \pi_G, \quad \text{(left invariant)} \\ \overrightarrow{H} &= \pi_G \cdot \mathbf{t}^*\Omega \cdot \pi_G. \quad \text{(right invariant)} \end{aligned}$$

(G, π_G): Poisson groupoid,

(AG, AG^*): the Lie bialgebroid,

$\mathbf{E}$: the double of (AG, AG^*)

<u>定理 B.</u>

$H \in \Gamma \bigwedge^2 AG$: $^\forall$Hamilton operator

$\Rightarrow$

(G, P_G): Poisson groupoid, where

$$P_G = \pi_G + \overleftarrow{H} - \overrightarrow{H}$$

そして, the Lie bialgebroid of (G, P_G)は(AG, L_H)に同型. 故に, そのManin tripleは

$$\{\tau_{-H}(\mathbf{E}), AG, AG^*\}.$$

つまり, $\{\mathbf{E}, AG, AG^*\}$のtwisting by H.

注意 B-1. (G_0, π_0)が現れていない.

例 A-3,B-2 (symplectic grouopidsのケース.)

(G, ω): symplectic groupoid,

$(M, \pi) := (G_0, \pi_0)$,

$u : AG \cong T^*M_\pi$.

この場合,

$$\Gamma \bigwedge^2 AG \ni H \cong \Omega \in \Gamma \bigwedge^2 T^*M_\pi.$$

そして,

$$H = u^{-1}(\Omega) = \sigma_*^* \cdot \Omega \cdot \sigma_*.$$

例 B-3 ($\pi_G = 0$のケース.)

このとき, $d_* = 0$.

故に, Hamilton operators H: $[H, H] = 0$ (CYBE).

$(G, \overleftarrow{H} - \overrightarrow{H})$: Poisson groupoid.

Z. Liu and P. Xu

Λ : r-matrix $\iff \mathfrak{L}_X[\Lambda, \Lambda] = 0$. $(\Lambda \in \Gamma \bigwedge^2 AG.)$

系 B-4 (generalized r-matrix.)

(G, π_G): Poisson grouopid with Λ, s.t.,

$$\mathfrak{L}_X(d_* \Lambda + \frac{1}{2}[\Lambda, \Lambda]) = 0$$

$\Rightarrow$

(G, P_G): Poisson groupoid, where

$$P_G := \pi_G + \overleftarrow{\Lambda} - \overrightarrow{\Lambda}.$$