Regulairzed Calculus and Its Applications

Akira Asada
Sinsyu University, 3-6-21 Nogami Takarazuka, 665-0022 Japan,
E-mail asada-a@poporo.ne.jp

Abstract

To overcome of the difficulty of divergence in the calculus on an nfinite dimensional space, a sytemaitc use of ζ-regularization was proposed nad named regularized calculus ([1]). In this paper, regularized calculus and some of its applications are reviewed.

1 Introduction

In the calculus on an infinite dimensional space, we soon meet the difficulty of divergence. For example, let H be a real Hilbert space with the coordinate $x = (x_1, x_2, \dots)$, and let $\triangle$ be the Laplacian

$$\triangle = \sum_{n=1}^{\infty} \frac{\partial^2}{\partial x_n^2}.$$

Then, even the metric function $r(x) = \|x\|$, $\triangle r(x)$ diverges. But if $H = L^2(X, E)$, the Hilbert space of square integrable sections of a bundle E over a compact Riemannian manifold X, taking a positive selfadjoint elliptic operator D acting on the sections of E, we can overcome this difficulty as follows: Taking the complete ortho-normal basis of H by eigenfunctions $e_1, e_2, \dots$; $De_n = \lambda_n e_n$, $0 < \lambda_1 \le \lambda_2 \le \dots$, of D, and set $x = \sum_n x_n e_n$, we introduce the operator $\triangle(s)$ by

$$\triangle(s) = \sum_{n=1}^{\infty} \lambda_n^{-2s} \frac{\partial^2}{\partial x_n^2}.$$

Then we define the regularized Laplacian $: \triangle :=: \triangle :_D$ by

$$: \triangle : f(x) = \triangle(s) f(x)|_{s=0},$$

where $|_{s=0}$ means analytic continuation to $s = 0$ ([1],[17]). By definition, we have

$$\begin{aligned} \triangle(s) r(x) &= \sum_{n=1}^{\infty} \frac{\lambda_n^{-2s}}{r(x)} - \sum_{n=1}^{\infty} \frac{\lambda_n^{-2s} x_n^2}{r(x)^3} \\ &= \frac{\zeta(D, 2s)}{r(x)} - \sum_{n=1}^{\infty} \frac{\lambda_n^{-2s} x_n^2}{r(x)^3}, \quad \zeta(D, s) = \sum_{n=1}^{\infty} \lambda_n^{-s}. \end{aligned}$$

Since X is compact, $\zeta(D,s)$ is holomorphic at $s=0$ ([11], cf.[7],[18]). Hence we obtain

$$:\triangle: r(x) = \frac{\nu - 1}{r(x)}, \quad \nu = \zeta(D,0).$$

Therefore $:\triangle: r(x)$ is defined.

This is an example of the application of ζ-regularization to overcome the difficulty of divergence in the calculus on an infinite diemnsional space. We proposed a systematic way of such applications called regularized calculus ([1]). In this paper, we review regularized calculus and some of its applications.

This paper is organized as follows: In §2, we introduce the pair $\{H, G\}$, H is a Hilbert space and G is a positive Schatten class operator such that $\zeta(G,s) = \mathrm{tr}G^s$ is holomorphic at $s=0$. Related spaces $H^\sharp$, *etc.* are also introduced. Throughout the paper, regularization is done by using spectres of G. The pair $\{H, G\}$ is closely related to Connes' spectral triple ([9]), and $H^\sharp \cong H \oplus \mathbb{R}$ may be interpreted as the (total space of the determinant bundle over H ([1],[5]). But we do not discuss on these points in this review.

Next two Sections deal with regularizations of elementary functions on $H^\sharp$. Especially, regularization of infinite product of coordinate functions of $H^\sharp$, which is named regularized infinite product, is precisely studied. If T is a scaling operator, regularized infinite product of eigenvalues of T can be regarded as a regularization of the determinant of T. Acording to this observation, we define regularized determinant of $T = \mathrm{e}^S$ by

$$\det{}_G T = \mathrm{e}^{\mathrm{tr}(G^s S)}|_{s=0},$$

(§5, cf.[2]). Since $\mathrm{tr}(G^s S)|_{s=0}$ is the renormalized trace of Paycha ([8],[15]), regularized determinant is an adaptation of renormalized trace.

§6 deals with regularized infinite dimensional integral. Since regularized determinant appears as the Jacobian of coordiante transform of regularized infinite dimensional integral, we obtain a mathematical justification of the appearence of Ray-Singer determinant in Physicists' calculation of Gaussian path integral ([2]). §7-§9 are devoted explanations of applications of regularized infinite integral, such as regularized volume form and regularized volume of infinite dimensional sphere ([3]), regularized Cauchy kernel([4]) and Fourier expansion of periodic functions of infinite variables ([6]). Then in §10, we solve periodic boundary value problem of regularized Laplacian, applying results on Fourier expansion of periodic functions on $H^\sharp$.

Existence of regularized volume form on spheres and possibility of Fourier expansion provide de Rham type cohomology with Poincaré duality derived from $H^\sharp$ (not from H). These examples suggest if a Hilbert manifold (or a manifold modeled limit space of Hilbert spaces) allows to define weak topology and compact by the weak toplogy, then the manifold has finite regularized volume. As a consequence, we can define de Rham type cohomology having Poincaré duality on such spaces (cf.[10],[13]). But at this stage, this is only feeling, not a conjecture. These are explained in $11, the last Section.

2 Hilbert space equipped with a Schatten class operator

Let H be a Hilbert space over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. G a positive Schatten class operator (cf.[16]) such that its ζ-function $\zeta(G, s) = \mathrm{tr}(G^s)$ is holomorphic at $s = 0$. Typical example of such pair is $H = L^2(X, E)$, the Hilbert space of square integrable sections of a vector bundle E over a compact Riemannian manifold X and G is the Green opertor of a positive selfadjoint elliptic pseudo differntial operator D acting on ths sections of E (cf.[11],cf.[7],[18]).

$\nu = \zeta(G, 0)$ and the location d of the first pole of $\zeta(G, s)$ are important numbers of the pair $\{H, G\}$. If G is the Green operator of m-th order elliptic equation on an n-dimensional compact Riemannina manifold, then $d = n/m$. This is the meaning of d. The residue of $\zeta(G, s)$ at $s = d$ is denoted by c. It is a positive real number.

Note. We often need integrity of ν. This is not restrictive in practice. Because, if G is the Green operator of an elliptic operator D on a compact Riemannina manifold, then

$$\zeta(D + mI, 0) = \zeta(D, 0) + \sum_{k=1}^{[d]} (-1)^k \mathrm{Res}_{s=k} \zeta(D, s) \frac{m^k}{k},$$

where $[d]$ is the Gauss' notation. Hence selecting m, $\zeta(D + mI, 0)$ becomes an integer ([1]).

We take eigenfunctions $e_1, e_2, \ldots$, $Ge_n = \mu_n e_n$, $\mu_1 \geq \mu_2 \geq \ldots > 0$, as the complete ortho-normal basis of H. Note that selection of $e_1, e_2, \ldots$ gives an additional structure to the pair $\{H, G\}$. But we have not yet any information obtained from the selection $e_1, e_2, \ldots$.

Since G is positive, the operator G^k; $G^k e_n = \mu_n^k e_n$, is uniquely defined. If $k < 0$, G^k is not defiend on H. Its domain is denoted by $\mathcal{D}(G^k)$.

Let $\|x\|$ and (x, y) be norm and inner product of H. Then we introduce Sobolev k-norm $\|x\|_k$, and inner product $(x, y)_k$, $x, y \in \mathcal{D}(G^{-k/2})$, by

$$\|x\|_k = \|G^{-k/2} x\|, \quad (x, y)_k = (G^{-k/2}, G^{-k/2}). \tag{1}$$

We denote W^k the completion of $\mathcal{D}(G^{-k/2})$ by this norm. By definitions, $H = W^0$ and

$$W^k = G^{-k/2} H, \quad G^{l/2} : W^k \cong W^{k-l}. \tag{2}$$

We set $e_{n,k} = G^{k/2}e_n = \mu_n^{k/2}e_n$. $\{e_{1,k}, e_{2,k}, \dots\}$ is a complete ortho-normal system of W^k.

Note. If $x = \sum_{n=1}^{\infty} x_{n,k}e_{n,k} \in W^{k,\sharp}$, the coordinate of $x \in W^{k,\sharp}$ is given by $(x_{1,k}, x_{2,k}, \dots)$. Then, since $e_{n,k} = \mu_n^{k/2}e_n$, we have $x_{n,k} = \mu_n^{-d/2}x_n$. Here $x = \sum_{n=1}^{\infty} x_n e_n$.

As sets, we have $W^k \subset W^l$, $k > l$. We set

$$\bigcap_{j<k} W^j = W^{k-0}, \quad \bigcup_{l>k} W^l = W^{k+0}. \tag{3}$$

If $k = 0$, we use notations $H^{\pm}$ instead of $W^{0\pm 0}$.

We say a series $x_1, x_2, \dots$ of elements of W^{k-0} converges to $x \in W^{k-0}$ if and only if $\lim_{n\to\infty} \|x_n - x\|_j = 0$ for all j; $j < k$. While a sereis $x_1, x_2, \dots$ of elements of W^{k+0} is said to converge to $x \in W^{k+0}$ if $\lim_{n\to\infty} \|x_n - x\|_l = 0$, for some l; $l > k$.

By definitions and (2), if $y \in W^{k+0}$, then we have $y = G^l y_0$, $y_0 \in W^k$, for some $l > 0$. On the other hand, if $x \in W^{k-0}$, we can set $x = G^{-l}x_0$, $x_0 \in W^k$ for any $l > 0$. Then we define the pairing $\langle x, y\rangle_k$ of $x \in W^{k-0}$ and $y \in W^{k+0}$ by

$$\langle x, y\rangle_k = (x_0, y_0)_k.$$

Since $W^{k-0} \subset W^{k-l} = G^{-l/2}W^k$, $l > 0$, as sets, we can express uniquely $x \in W^{k-0}$ as $\sum_{n=1}^{\infty} x_n e_{n,k}$. By using this expression, we set

$$e_{\infty,k} = \sum_{n=1}^{\infty} \mu_n^{d/2} e_{n,k}. \tag{4}$$

If $k = 0$, we denote e_∞, instead of $e_{\infty,0}$. By definition, $e_{\infty,k} \in W^{k-0}$ but does not belong to W^k.

Definition 1. *We set*

$$W^{k,\sharp} = W^k \oplus \mathbb{K}e_{\infty,k} \subset W^{k-0}. \quad H^\sharp = W^{0,\sharp}. \tag{5}$$

Note. $W^{k,\sharp}$ does not determined by the pair $\{H, g\}$. It depends on the choice of eigenfunctions $e_1, e_2, \dots$ of G.

We consider $W^{k,\sharp}$ to be a subspace of W^{k-0}. Hence it is not a Hilbewrt space. But since $W^{k,\sharp} = W^k \oplus \mathbb{K}e_{\infty,k}$, if $x \in W^{k,\sharp}$, then it is written uniquely

$$x = x_f + te_\infty, \quad x_f \in W^k, \ t \in \mathbb{K},$$

we can introduce an inner product $(x, y)_{k,\natural}$ of $x, y \in W^{k,\sharp}$ by

$$(x, y)_{k,\natural} = \lim_{s\downarrow 0}(G^{s/2}(x_f + \sqrt{s}te_{\infty,k}), G^{s/2}(y_f + \sqrt{s}ue_{\infty,k})_k.$$

$W^{k,\sharp}$ becomes a Hilbert space by this inner product $W^{k,\natural}$. A complete orthonormal basis of this space is given by $\{e_{1,k}, e_{2,k}, \dots, (\sqrt{c})^{-1}e_{\infty,k}\}$.

There is an alternative way to extend Hilbert space, based on polar coordinate of H. Denoting $r = \|x\|$, $x = \sum_n x_n e_n$, polar coordinate of H is defined by

$$\begin{aligned} x_1 &= r\sin\theta_1,\ x_2 = r\cos\theta_1\sin\theta_2, \dots, \\ x_n &= r\cos\theta_1\cdots\cos\theta_{n-1}\sin\theta_n, \dots . \quad 0 \le \theta_n \le \pi. \end{aligned}$$

This polar coordiante has no longitude. Latitudes are not independent. They need to satisfy the constraint

$$\lim_{n\to\infty} \sin\theta_1\cdots\sin\theta_n = 0. \tag{6}$$

Putting out of this constraint, a new variable $t_\infty = \pm\prod_{n=1}^{\infty}\sin\theta_n$ is obtianed. Adding this variable to H, we obtain an extension $\hat{H}$ of H.

Polar coordiante is also defined on W^s. Denoting latitudes of $e_\infty \in W^{-s}$, $s > 0$, by $\theta_n(s)$, we have

$$\theta_n(s) = \frac{\pi}{2} - \frac{\mu_n^{d/2}}{\sqrt{c}}\sqrt{s} + O(s),\ s \downarrow 0,$$

([5]). Hence by the polar coordiante of $\hat{H}$, we have

$$\lim_{s\downarrow 0}\sqrt{s}e_\infty = (\sqrt{c}, \frac{\pi}{2}, \frac{\pi}{2}, \dots).$$

Therefore, we have ([5])

Proposition 1. *By the map* ρ: $\rho(x, te_\infty) = (x, t\sqrt{c}) \in \hat{H}$, *we can identify* $H^\natural$ *and* $\hat{H}$.

3 Elementary functions on $H^\sharp$

Even the sum $sigma_1(x) = \sum_{n=1}^{\infty} x_n$ of coordinates of $x \in H$ can not define on H. So we regularize $\sigma_1(x)$ as follows;

$$:\sigma_1(x) := \sum_{n=1}^{\infty}\mu_n^s x_n|_{s=0}.$$

For example, if $x = x(t) = \sum_{n=1}^{\infty}\mu_n^t e_n$, $t > d/2$, then $\sigma_1(x(t))$ defined for $t > d$, but $:\sigma(x(t)):$ is defiend for $t > d/2$, $t \ne d, d_1, \dots, d_m$, $d > d_1 > \cdots d_m \ge d/2$, where $d_1, \dots, d_m$ are poles of $\zeta(G, s)$. We note that if we introduce the scaling operator I_x; $I_x e_n = x_n e_n$, we have

$$:\sigma_1(x) := \mathrm{tr}(G^s I_x)|_{s=0}. \tag{7}$$

Hence $:\sigma_1(x):$ is a kind of renormalized trace of Paycha ([8],[15]). By (7), if $:\sigma_1(ax+by):$, $a,b\in\mathbb{K}$, and $:\sigma_1(x):$, $:\sigma_1(y):$ both exist, then we have

$$:\sigma_1(ax+by):=a:\sigma_1(x):+b:\sigma_1(y):. \tag{8}$$

Regularization of the k-th order elementary symmetric function $\sigma_k(x)=\sum_{i_1,\dots,i_k}x_{i_1}\cdots x_{i_k}$ is defined by

$$:\sigma_k(x):=\sum_{i_1,\dots,i_k}\mu_{i_1}^s\cdots\mu_{i_k}^s x_{i_1}\cdots x_{i_k}|_{s=0}.$$

By definitions, we have

$$\prod_{n=1}^{\infty}(1+\mu_n^s x_n t)|_{s=0}=1+\sum_{k=1}^{\infty}:\sigma_k(x):t^k. \tag{9}$$

Since we have

$$\log(\prod_{n=1}^{\infty}(1+\mu_n^s x_n t)=\sum_{n=1}^{\infty}\log(1+\mu_n^s x_n t)$$
$$=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\mu_n^{ks}x_n^k=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sum_{n=1}^{\infty}\mu_n^{ks}x_n^k,$$

and since $\sum_{n=1}^{\infty}x_n^k$ converges if $x\in H$ and $k\geq 2$, we obtain

Proposition 2. *If $x\in H$ and $:\sigma_1(x):$ exists, then $:\sigma_k(x):$ exists for all k; $k\geq 2$.*

We note that since $:\sigma_1(e_\infty):=\zeta(G,d/2)$, $:\sigma_1(e_\infty):$ exists, if $\zeta(G,s)$ is holomorphic at $s=d$. But $:\sigma_2(e_\infty):$ does not exists, because $\zeta(G,s)$ has a pole at $s=d$. Hence this Proposition does not hold on $H^\sharp$.

Note. $\prod_{n=1}^{\infty}x_n$ does not appear in the expansion of $\prod_{n=1}^{\infty}(1+x_nt)$. Its regularization is discussed in the next Section.

$:\sigma_j(x):$ is also defiend for $x\in W^k$ or $x\in W^{k,\sharp}$. But since $W^k=G^{k/2}H$, $\sigma_j(x)$ exists if $k>2d$, so we need not use regularization to consider elementary symmetric functions if $k>2d$.

Next we consider infinite product of trigonometric functions

$$s_{n_m}=\sin 2n_m\mu_n^{-d/2}x_n,\quad c_{n_m}=\cos 2n_m\mu_n^{-d/2}x_n.$$

Here, $\lim_{n\to\infty}n_m=n_\infty$ must exists, because $\sum_n n_m x_n e_n\in H^\sharp$, for any $0\leq x_n\leq\mu_n^{d/2}$. Under this assumption, the infinite product $\prod_{n=1}^{\infty}f_n(x_n)$, $f_n(x_n)$ is either of s_{n_m}, or c_{n_m}, does not vanish on $H^\sharp$ if and only if $f_n(x_n)=s_{n_m}$

except finite numbers $n(1),\dots,n(j)$ of n, or $f_n(x_n) = c_{n_m}$, except finite numbers $n(1),\dots,n(j)$ of n. Here we do not need regularization.

On the other hand, $\prod_{n=1}^{\infty} f_n(x_n) = 0$ on H, unless $f_n(x_n) = 1$, except finite numbers $n(1),\dots,n(j)$ of n.

We conclude this Section to add some remarks on Fréchet differentiable functions on U, a subset of $H^\sharp$. If a function f on U is Fréchet differentiable, then for some $j < 0$, we have $f(x+ty) = f(x) + t(df(x),y)_j + o(t)$, and $\|y\|_j < \infty$. Hence we have

Lemma 1. *Let f be a Fréchet differentiable function on a subset of $H^\sharp$, then*

$$\|f(x) - f(x_n\|_l \leq (1+\epsilon)\|df(x)\|_j\|x - x_n\|_j, \quad \epsilon > 0, \tag{10}$$

for some $j < 0$, if n is large.

4 Regularized infinite product

Let $x_1, x_2, \dots$ be a series of complex number having an Agmon angle θ; $\theta < \mathrm{Arg} x_n < \theta + 2\pi$. Then we define regularized infinite product : $\prod_{n=1}^{\infty} x_n$:=: $\prod_{n=1}^{\infty} x_n :_{G,\theta}$ $(- : \prod_{n=1}^{\infty} x_n :_G)$, by

$$: \prod_{n=1}^{\infty} x_n :_{G,\theta} = \prod_{n=1}^{\infty} x_n^{\mu_n^s}|_{s=0}, \quad x_n^{\mu_n^s} = |x_n|^{\mu_n^s} \mathrm{e}^{\mu_n^s \mathrm{i} \mathrm{Arg} x_n}. \tag{11}$$

Note that if $\sum_{n=1}^{\infty} x_n e_n \in H^\sharp$, then $\{x_1, x_2, \dots\}$ has an Agmon angle if $x \notin H$. Because to set $x_n = x_{n,f} + t\mu_n^{d/2}$,$t \neq 0$, we have

$$\lim_{n\to\infty} \mathrm{Arg} x_n = \mathrm{Arg} t.$$

Example. If $x_n = \mu_n^k$, then

$$\begin{aligned} : \prod_{n=1}^{\infty} \mu_n^k :_G &= \prod_{n=1}^{\infty} \mu_n^{k\mu_n^s}|_{s=0} = \prod_{n=1}^{\infty} \mathrm{e}^{k\mu_n^s \log \mu_n}|_{s=0} \\ &= \mathrm{e}^{\sum_{n=1}^{\infty} k \log \mu_n \mu_n^s}|_{s=0} = \mathrm{e}^{k\zeta'(G,s)}|_{s=0} = (\det G^k). \end{aligned}$$

Here $\det G^k$ means Ray-Singer determinant of G^k.

By definition, as a function of $x_1, x_2, \dots$, : $\prod_{n=1}^{\infty} x_n$: is linear in each variable, and we have

$$;\prod_{n=1}^{\infty} x_n^k :_{G,k\theta} \; = \; (:\prod_{n=1}^{\infty} x_n :_{G,\theta})^k, \quad k\in\mathbb{R}, \tag{12}$$

$$:\prod_{n=1}^{\infty} tx_n :_{G,\theta} \; = \; t^{\nu} :\prod_{n=1}^{\infty} x_n :_{G,\theta}, \quad t>0, \tag{13}$$

$$:\prod_{n=1}^{\infty} |x_n| :_{G} \; = \; |\prod_{n=1}^{\infty} x_n :_{G} |. \tag{14}$$

$$:\prod_{n=1}^{\infty} x_n y_n :_{G,\theta_1+\theta_2} \; = \; :\prod_{n=1}^{\infty} x_n :_{G,\theta_1} \cdot :\prod_{n=1}^{\infty} y_n :_{G,\theta_2} . \tag{15}$$

$$:\prod_{n=1}^{\infty} G^k x_n :_{G} \; = \; (\det G)^k :\prod_{n=1}^{\infty} x_n :_{G} . \tag{16}$$

Therefore $:\prod_{n=1}^{\infty} x_n :$ is differentiable in each x_n and we may consider

$$\frac{\partial^N}{\partial x_{i_1}\cdots\partial x_{i_N}} :\prod_{n=1}^{\infty} x_n :=: \prod_{n\notin\{i_1,\dots,i_N\}} x_n : . \tag{17}$$

If $x = x_f + te_\infty \in H^\sharp$, $t\neq 0$, $x_f = \sum_n x_{f,n}e_n \in H$, then

$$\begin{aligned} :\prod_{n=1}^{\infty} x_n : \; &= \; \prod_{n=1}^{\infty}(x_{f,n} + t\mu_n^{d/2})^{\mu_n^s}|_{s=0} \\ &= \; t^{\zeta(G,s)}\big(\prod_{n=1}^{\infty} \mu_n^{\mu_n^s}\big)^{d/2} \prod_{n=1}^{\infty}\big(1+\frac{x_{f,n}}{t\mu_n^{d/2}}\big)^{\mu_n^s}|_{s=0} \\ &= \; t^{\nu}(\det G)^{d/2}\prod_{n=1}^{\infty}\big(1+\frac{x_{f,n}}{t\mu_n^{d/2}}\big), \end{aligned}$$

if $\sum_{n=1}^{\infty}|\mu_n^{-d/2}x_{f,n}|<\infty$. Hence denoting

$$\ell^{1,c} = \{\sum_{n=1}^{\infty} x_n e_n | \sum_{n=1}^{\infty}\mu_n^{-c}|x_n| < \infty\},$$

$:\prod_{n=1}^{\infty} x_n :$ is defined on $\ell^{1,d/2}\oplus\mathbb{K}^\times e_\infty$. Here $\mathbb{K}^\times$ means $\mathbb{K}\setminus\{0\}$. This function is single valued if and only if ν is an integer.

Let $x = x_f + te_\infty = \sum_n x_n e_n \in H^\sharp$ and let $\check{x} = te_\infty - x_f = \sum_n \check{x}_n e_n$, then $:\prod_{n=1}^{\infty} x_n : \cdot : \prod_{n=1}^{\infty}\check{x}_n :$ exists if $x_f \in W^d$. Therefore the function $:\prod_{n=1}^{\infty} x_n :$ is analytic on $W^d\oplus\mathbb{K}e_\infty \subset H^\sharp$.

Similarly, by (16), if we consider $:\prod_{n=1}^{\infty} x_n :$ to be a function on $W^{k,\sharp}$, it is defined on $\ell^{1,(k+d)/2}\oplus\mathbb{K}^\times e_{\infty,k}$ and analytic on $W^{k+d}\oplus\mathbb{K}^\times e_{\infty,k}$.

Note. In these calculations, we need to assume $x \notin W^k$. But since

$$: \prod_{n=1}^{\infty}(t+x_n) := t^{\nu} \prod_{n=1}^{\infty}\big(1+\frac{x_n}{t}\big) :,$$

we may define finite part of $: \prod_{n=1}^{\infty} x_n :$ by

$$\text{p.f.} : \prod_{n=1}^{\infty} x_n :=: \sigma_{\nu}(x) :, \tag{18}$$

if ν is a positive integer.

For a series $x = (x_1, x_2, \dots)$, we set $\log x = (\log x_1, \log x_2, \dots)$. Then we have $\log I_x = I_{\log x}$, that is we have

$$\mathrm{e}^{I_{\log x}} = I_x.$$

Then, since

$$\begin{aligned} \mathrm{e}^{\mathrm{tr}(G^s I_{\log x})}|_{s=0} &= \mathrm{e}^{\sum_{n=1}^{\infty} \mu_n^s \log \mu_n}|_{s=0} \\ &= \prod_{n=1}^{\infty} x_n^{\mu_n^s}|_{s=0} =: \prod_{n=1}^{\infty} x_n :_G, \end{aligned}$$

we have

Lemma 2. *We have*

$$: \prod_{n=1}^{\infty} x_n :_G = \mathrm{e}^{\mathrm{tr} G^s I_{\log x}}|_{s=0}. \tag{19}$$

5 Regularized determinant

Suggested by Lemma 2, we define regularized determiannt of a linear operator T as follows;

Definition 2. *Let T be a linear operator densely defined on H. If T has the logarithm $\log T = S$; $\mathrm{e}^S = T$, then we define regularized determnant $\det_G T$ of T with respect to G by*

$$\det{}_G T = \mathrm{e}^{\mathrm{tr}(G^s S)}|_{s=0}. \tag{20}$$

By Lemma 2 and Example of §4, we have

$$\det{}_G I_x =: \prod_{n=1}^{\infty} x_n :_G, \quad \det{}_G G = \det G.$$

But since $\log T$ is not unique, $\det_G T$ is not unique in general. For example, we have

$$\det G(tI) = \mathrm{e}^{\log t \zeta(G,s)}|_{s=0} = t^{\nu}.$$

If ν is not an integer, t^{ν} is not unique.

If $\not{D}$ is the Dirac operator (assumed to have no 0-mode), and G is the Green operator of $|\not{D}|$, then

$$\det_G \not{D} = (-1)^{\nu_-} \det|\not{D}|, \quad \nu_- = \frac{\zeta(|\not{D}|, 0) - \eta(\not{D}, 0)}{2}.$$

Hence $\det_G \not{D}$ is not unique unless ν_- is an integer ([1]).

If $T = I + U$, U a trace class operator, then $\det(I + U)$ is defined ([16]). In this case, we can determine $\log(I + sU)$, $\|sU\| < 1$, by

$$\log(I + sU) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} s^n U^n, |s| < \|U\|^{-1}.$$

Hence $\det_G(I + sU) = \det(I + sU)$ if $|s| < \|U\|^{-1}$. Then by analytic continuation, we have $\det_G(I + U) = \det(I + U)$.

If $T_1 T_2 = T_2 T_1$ and $\log T_1 \log T_2 = \log T_2 \log T_1$, then we have

$$\det_G(T_1 T_2) = \det_G T_1 \det_G T_2.$$

Especially, we have

$$\det_G(tT) = t^{\nu} \det_G T. \tag{21}$$

Lemma 3. *If both* $T_1 = \mathrm{e}^{S_1}$, $T_2 = \mathrm{e}^{S_2}$ *have regularized determinants and Campbell-Haussdorff′formula is valid for* T_1^t, T_2^t *and moreover* $G^s S_2 = S_2 G^s$ *is hold, then we have* $\det_G T_1 T_2 = \det_G T_1 \mathrm{dt}_G T_2$.

Proof. By assumptions, we have $\mathrm{tr} G^s S_2 H = \mathrm{tr} S_2 G^s H = \mathrm{tr} G^s H S_2$, for any H. Hence $\mathrm{tr} G^s [S_2, H]$ is equal to 0. On the other hand, by Campbell-Haussdorff' formula (cf.[14]), we have

$$T_1^t T_2^t = \mathrm{e}^{tS_1} \mathrm{e}^{tS_2} = \mathrm{e}^{t(S_1+S_2)+[S_2, f(t,S_1,S_2)]},$$

if $|t|$ is small. Hence we obtain

$$\begin{aligned} \det_G(T_1^t T_2^t) &= \mathrm{e}^{\mathrm{tr}(G^s \log(T_1^t T_2^t)}|_{s=0} = \mathrm{e}^{\mathrm{tr}(G^s(t(S_1+S_2)+[S_2, f(t,S_1,S_2)])}|_{s=0} \\ &= \mathrm{e}^{\mathrm{tr}(G^s(t(S_1+S_2)))}|_{s=0} = \mathrm{e}^{\mathrm{tr}(tG^s S_1)}|_{s=0} \mathrm{e}^{\mathrm{tr}(tG^s S_2)}|_{s=0}. \end{aligned}$$

Therefore we have Lemma by analytic continuation with respect to t.

We have $\det_G(PTP^{-1}) = \det_{P^{-1}GP}T$. But it may different from $\det_G T$. For example, if

$$\begin{cases} Ge_{2n-1} = \frac{1}{n}e_{2n-1}, \\ Ge_{2n} = \frac{1}{n+1}e_{2n}, \end{cases} \quad \begin{cases} Te_{2n-1} = 3e_{2n-1}, \\ Te_{2n} = 2e_{2n}, \end{cases} \quad \begin{cases} Pe_{2n-1} = e_{2n}, \\ Pe_{2n} = e_{2n-1}, \end{cases}$$

then we have

$$\det_G T = 3^{\zeta(s)}2^{\zeta(s)-1}|_{s=0} = \frac{1}{2\sqrt{6}} \neq \det_G PTP^{-1} = 2^{\zeta(s)}3^{\zeta(s)-1} = \frac{1}{3\sqrt{6}}.$$

If $PG = GP$, or $P = I + K$, K is a compact operator, then we have $\det_G T = \det_G(PTP^{-1})$ for any T. To apply regularized infinite dimensional integral, P need not be bounded. But it needs to map $H^\sharp$ (or $W^{k,\sharp}$) into $W^{l,\sharp}$. Determination of the group of such operators is a future problem.

To extend the formula $\det_G G^k = \mathrm{e}^{k\zeta'(G,0)}$, we first note

$$\frac{d^m}{dt^m}G^t = (\log G)^m G^t$$

is hold if we consider $\{G^t | t \in \mathbb{R}\}$ to be the 1-parameter group of operatros on $W^\infty = \cup_l W^l$. We also define the operator $\mathrm{e}^{t(\log G)^k G^m}$, $k \in \mathbb{N}$, $m \in \mathbb{C}$, where $\zeta(G,s)$ is holomorphic at m, by

$$\mathrm{e}^{t(\log G)^k G^m} e_n = \mathrm{e}^{t(\log \mu_n)^k \mu_n^m} e_n.$$

By definition, if $k \geq 1$, we have

$$\mathrm{e}^{t(\log G)^k G^m} = G^{t(\log G)^{k-1} G^m}.$$

Proposition 3. *If $\zeta(G,s)$ is holomorphic at $s = m$, then*

$$\det_G \mathrm{e}^{t(\log G)^k G^m} = \mathrm{e}^{t\zeta^{(k)}(G,m)}, \tag{22}$$

$$\det_G G^{t(\log G)^{k-1} G^m} = \mathrm{e}^{t\zeta^{(k)}(G,m)}, \quad k \geq 1. \tag{23}$$

Proof. Since $\frac{d}{ds}\mathrm{tr}G^{m+s} = \mathrm{tr}\big(\frac{d}{ds}G^{m+s}\big)$, we have

$$\begin{aligned} \mathrm{e}^{t\zeta^{(k)}(G,m+s)}|_{s=0} &= \mathrm{e}^{t\frac{d^k}{ds^k}(\mathrm{tr}G^{m+s})}|_{s=0} = \mathrm{e}^{t\mathrm{tr}(\frac{d^k}{ds^k}G^{m+s})}|_{s=0} \\ &= \mathrm{e}^{t\mathrm{tr}((\log G)^k G^{m+s})}|_{s=0} = \det_G \mathrm{e}^{t(\log G)^k G^m}. \end{aligned}$$

For example, if G is the Green operator of an elliptic operator D, then the heat kernel $\mathrm{e}^{-tD} = \mathrm{e}^{-tG^{-1}}$ has the regualrized determinat with respect to G if and only if $\zeta(G,s)$ is holomorphic at $s=-1$, and we have

$$\det{}_G \mathrm{e}^{-tD} = \mathrm{e}^{-t\zeta(G,-1)}.$$

Hence $\det_G \exp(-tD)$ is independent of t if $\zeta(G,-1)=0$.

In genral, by Lemma 3, if $\det_G C$ exists and Campbell-Haussdorff' formula hold for C^t and e^{-tD}, then

$$\det{}_G(C\mathrm{e}^{-tD}) = \det{}_G C \cdot \det{}_G \mathrm{e}^{-t\zeta(G,-1)}. \tag{24}$$

It is also seen, if $\sum_{n=1}^{\infty}(1+|x_n|) < \infty$, then

$$:\prod_{n=1}^{\infty} \mathrm{e}^{t(\log \mu_n)^k \mu_n^m}(1+x_n) := \mathrm{e}^{t\zeta^{(k)}(G,m)} \prod_{n=1}^{\infty}(1+x_n),$$

if $\zeta(Gs)$ is holomorphic at $s=m$.

Note. The proof of Proposition 3 also shows if $\zeta(G,s)$ has a pole at $s=m$, then some regularized infinite products with respect to G, such as $:\prod_{n=1}^{\infty} \mathrm{e}^{\mu_n^m} :_G$, do not exist.

If m is negative, $x(m) = \sum_{n=1}^{\infty} \mathrm{e}^{-\mu_n^m} e_n \in H$. Hence if $\zeta(G,s)$ has a pole at $s=m$, the function $:\prod_{n=1}^{\infty} x_n$ has singularity at $x = x(m)$. This showes $:\prod_{n=1}^{\infty} x_n :$ may have many singularities on H (or on $H^\sharp$).

6 Regularized infinite dimensional integral

In this Section, we assume H is a real Hilbert space. Let $a = (a_1, a_2, \dots)$ and $b = (b_1, b_2, \dots)$ be serieses of real numbers or $\pm\infty$ such that $a_n < b_n$, $n = 1, 2, \dots$ and $\lim_{n\to\infty} \mu_n^{-d/2} = t$, $\lim_{n\to\infty} \mu_n^{-d/2} b_n = \bar{t}$ both exists. Here t and $\bar{t}$ may take $-\infty$ or ∞. Then we define $\mathcal{D}_{a,b} \subset H^\sharp$ by

$$\mathcal{D}_{a,b} = \{\sum_{n=1}^{\infty} x_n e_n \in H^\sharp | a_n \le x_n \le b_n\}. \tag{25}$$

By definition, we have

$$G^{k/2}\mathcal{D}_{a,b} = \{\sum_{n=1}^{\infty} x_n e_{n,k} \in W^{k,\sharp} | a_n \le x_n \le b_n\} (= \mathcal{D}_{a,b;k}). \tag{26}$$

We regard $\mathbb{R}^N$ to be $\{\sum_{n=1}^{N} x_n e_n \in H^\sharp\}$, and set

$$\mathcal{D}_{a,b}^N = \{\sum_{n=1}^{N} x_n e_n | a_n \le x_n \le b_n\} \subset \mathbb{R}^N.$$

We also take a basepoint $\xi = (\xi_1, \xi_2, \dots) \in \mathcal{D}_{a,b}$ and set

$$\xi = (\xi_1, \dots, \xi_N, \xi^N); \quad \xi^N = (\xi_{N+1}, \xi_{N+2}, \dots).$$

Definition 3. *We define the regularized integral of a function f on $\mathcal{D}_{a,b}$ by*

$$\int_{\mathcal{D}_{a,b}} f(x) : d^\infty x := \lim_{N\to\infty} \int_{\mathcal{D}_{a,b}^N} f(x_1, \dots, x_N, \xi^N) d(x_1^{\mu_1^s}) \cdots d(x_N^{\mu_N^s})|_{s=0}. \quad (27)$$

By Lemma 1, if f is Fréchet differentiable, this limit deos not depend on the choice of ξ, if it exists.

Note. If $a_n < 0$, we need to fix its argument. For the simplicity, such selection must be done simultaneously for all coordinates. Hence there are two kinds of regularized infinite dimensional integral in this case, according to the selection $-1 = \mathrm{e}^{\mathrm{i}\pi}$, or $-1 = \mathrm{e}^{-\mathrm{i}\pi}$.

Regularized integral of the function on $\mathcal{D}_{a,b;k}$ is similarly defined. Denoting regularized volume form on $\mathcal{D}_{a,b;k}$ by $: d^\infty x :_k$, we have

$$: d^\infty x :_k = (\det G)^{-k/2} : d^\infty x : . \quad (28)$$

Because $d(x_{n,k}^{\mu_n^s}) = \mu_n^{(-k/2)\mu_n^s} d(x_n^{\mu_n^s})$. In general, since

$$\int_a^b f(x) d(x^{\mu^s}) = \int_{a/c}^{b/c} f(cy) c^{\mu^s} d(y^{\mu^s}), \quad x = cy,$$

we have ([2])

Theorem 1. *Let I_c be the scaling operator $I_c e_n = c_n e_n$, $c_1 > 0, c_2 > 0, \dots$, then*

$$\int_{\mathcal{D}_{a,b}} f(x) : d^\infty x := \int_{I_c(\mathcal{D}_{a,b})} (: \prod_{n=1}^\infty c_n :)^{-1} f(I_c^{-1} x) : d^\infty (I_c x) : . \quad (29)$$

Corollary. *If $f(x) = \prod_{n=1}^\infty f_n(x_n)$, and $\int_{a_n}^{b_n} f_n(x_n) dx_n = c_n > 0$, then*

$$\int_{\mathcal{D}_{a,b}} f(x) : d^\infty x :=: \prod_{n=1}^\infty \int_{a_n}^{b_n} f_n(x_n) dx_n : . \quad (30)$$

Proof. If $\int_a^b f(x)dx = c \neq 0$, then to set $y = cx$, we have $\int_{a/c}^{b/c} f(y)dy = 1$. Hence by using scaling transformation I_c; $I_c e_n = c_n e_n$, we have Corollary.

Example. Let G be the Green operator of a positive elliptic operator D. Then

$$\int_{W^{1/2,\sharp}} \mathrm{e}^{-\pi(x,Dx)} : d^\infty x := \frac{1}{\sqrt{\det D}}.$$

Here $\det D$ is the Ray-Singer determinant of D ([2]).

The assumption $a_n < b_n$ in Definition 3 can be relaxed as follows: Let $\sigma = (\pm_1, \pm_2, \dots)$ be a sereis of symbols $\pm$ and let

$$\int_{\mathcal{D}^N_{a,b:\sigma}} f(x) d^N x \quad = \quad \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_N}^{\beta_n} f(x) d^N x,$$

Here $\alpha_n = a_n$, $\beta_n = b_n$, if $\pm_n = +$ and $\alpha_n = b_n$, $\beta_n = a_n$, if $\pm_n = -$. Then $\int_{\mathcal{D}_{a,b:\sigma}} f(x) : d^\infty x :$ is defined if $\pm_n = +$ except finite m-numbers, or $\pm_n = -$ except finite m-numbers. In these cases, we have

$$\int_{\mathcal{D}_{a,b:\sigma}} f(x) : d^\infty x := (-1)^{n(\sigma)} \int_{\mathcal{D}_{a,b}} f(x) : d^\infty x : . \tag{31}$$

Here $n(\sigma) = m$ if most of $\pm_n$ are $+$ and $\nu - m$ if most of $\pm_n$ are $-$. Note that if ν is not an integer, $(-1)^{\nu - m}$ depends on the choice of the argument of -1.

As a consequence, Theorem 1 and its Corollary are valid for the scaling operator I_c, where $c_1 \neq 0, c_2 \neq 0, \dots$, and they have same sign except finite $c_{i_1}, \dots, c_{i_m}$, or for $f(x) = \prod_{n=1}^\infty f_n(x_n)$ where $\int_{a_n}^{b_n} f_n(x_n) dx_n \neq 0$, $n = 1, 2, \dots$, and they have same sign except finit $n_1, \dots, n_m$.

Note. (29) suggests general formula

$$\int_{F(\mathcal{D}_{a,b})} f(F(x)) : d^\infty F(x) := \int_{\mathcal{D}_{a,b}} |\det{}_G dF| f(x) : d^\infty x :,$$

might be hold. If we get such formula, we can combine regularized integral and probalistic integral.

But since rgularized determinant is not invariant under the conjugation by inversible operator, it seems to get this type equality, we need to clarify the group of operators whose conjugations preserve regularized determinant.

We also note that by (30), we can define regularized volume of $\mathcal{D}_{a,b}$ by

$$: \mathrm{vol}(\mathcal{D}_{a,b}) := \int_{\mathcal{D}_{a,b}} : d^\infty x :=: \prod_{n=1}^\infty (b_n - a_n) : . \tag{32}$$

By definition, $: \mathrm{vol}(\mathcal{D}_{a,b}) :$ is translation invarinat. But we can not derive a nontrivial measure on the family of Borel sets generated by open sets of $H^\sharp$

by using regularized volume. Because there exists a, b such that $:\mathrm{vol}(\mathcal{D}_{a,b}):$ does not exists by (22).

On th other hand, it seems there are possibilities to give Riemannian integral type definitin to regularized integral by using regularized volume. This might be a future porblem.

There are two alternative definitions of regularized integral. The first definition uses fractional integral

$$I^{\mu}_{x} f(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t) dt.$$

Then, since

$$\log\big(\prod_{n=1}^{\infty} \Gamma(1+\mu_n^s)\big) = -\gamma\zeta(G,s) + \sum_{m=2}^{\infty} (-1)^m \frac{\zeta(m)}{m} \zeta(G,ms),$$

we have $\prod_{n=1}^{\infty} \Gamma(1+\mu_n^s)|_{s=0} = 1$. Hence we get

$$\int_{\mathcal{D}_{0,x}} f(x) : d^{\infty}x := \lim_{N\to\infty} I^{\mu_1^s}_{x_1} \cdots I^{\mu_N^s}_{x_N} f(x_1, \dots, x_N, \xi^N)|_{s=0}. \tag{33}$$

The second definition uses suitabel function space on which

$$\lim_{N\to\infty} \frac{\partial^N}{\partial x_1 \cdots \partial x_N} : \prod_{n=1}^{\infty} x_n := 1,$$

is hold in the weak sense (cf.[12]). Then using same function space, we have

$$\begin{aligned} &\lim_{N\to\infty} \int_{\mathcal{D}^N_{a,b}} \big(\frac{\partial^N}{\partial x_1 \cdots \partial x_N} x_1^{\mu_1^s} \cdots x_N^{\mu_N^s}\big) f(x_1, \dots, x_N, \xi^N) d^N x|_{s=0} \\ = &\int_{\mathcal{D}_{a,b}} f(x) : d^{\infty}x :, \end{aligned} \tag{34}$$

in the weak sense on this function space. An example of such function space is $L^{2,1}_b(H^{\sharp})$, the Sobolev 1-type L^2-space generated by

$$f(x_1, \dots, x_n) \mathrm{e}^{-\pi(\sum_{m>n} x_m^2)}, \quad f \in C^1_b(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),$$

([2]). Here $C^1_b(\mathbb{R}^n)$ is the space of bounded C^1-class functions on $\mathbb{R}^n$ and we set

$$\mathrm{e}^{-\pi(\sum_{m>n} x_m^2)} = 0, \text{ if } \sum_{m>n} |x_m|^2 = \infty,$$

7 Regularized integral by polar coordinate

Let r be $\|x\|$, $x \in \mathbb{R}^N$. Then by the polar coordinate

$$\begin{aligned} x_1 &= r\cos\theta_1, \dots, x_{N-2} = r\sin\theta_1 \cdots \sin\theta_{N-3}\cos\theta_{N-2}, \\ x_{N-1} &= r\sin\theta_1 \cdots \sin\theta_{N-2}\cos\phi,\ x_N = r\sin\theta_1\cdots\sin\theta_{N-2}\sin\phi, \end{aligned}$$

of $\mathbb{R}^N \subset H^\sharp$, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \big(\frac{\partial^N}{\partial x_1 \cdots \partial x_N} x_1^{\mu_1^s} \cdots x_N^{\mu_N^s}\big) f(x) d^N x \\ = &\int_0^\infty \int_{S^{N-1}} r^{\mu_1^s + \cdots \mu_N^s - N} \times \\ &\times \sin^{\mu_2^s + \cdots + \mu_N^s - N+1}\theta_1 \cdots \sin^{\mu_{N-1}^s + \mu_N^s}\theta_{N-2} F_N(x;s) f(x) \times \\ &\times r^{N-1}\sin^{N-2}\theta_1 \cdots \sin\theta_{N-2} d\phi \\ = &\int_0^\infty r^{\mu_1^s + \cdots + \mu_N^s - 1} dr \int_{S^{N-1}} F_N(x;s) f(x) \times \\ &\times \sin^{\mu_2^s + \cdots \mu_N^s - 1}\theta_1 d\theta_1 \cdots \sin^{\mu_{N-1}^s + \mu_N^s - 1}\theta_{N-2} d\theta_{N-2} d\phi. \end{aligned}$$

Here $F_N(x;s) = \mathcal{F}_N(|\cos\theta_1|, \dots, |\cos\theta_{N-2}|, |\cos\phi|, |\sin\phi|; s)$, where

$$\mathcal{F}(x;s) = \frac{\partial^N}{\partial x_1 \cdots \partial x_N} x_1^{\mu_1^s} \cdots x_N^{\mu_N^s}.$$

Since $\lim_{N\to\infty} \mathcal{F}_N(x;s) = 1$ by the weak topology of $W^1(\mathbb{R}^\infty)$, we have

$$F(\theta_1, \theta_2, \dots; s)|_{s=0} = 1, \quad F(\theta_1, \theta_2, \dots; s) = \lim_{N\to\infty} F_N(x;s)|_{\phi=0,\pi}.$$

Hence by the weak toplogy of $L_b^{2,1}(H^\sharp)$, we have

$$\begin{aligned} &\lim_{N\to\infty} \int_{\mathbb{R}^N} \big(\frac{\partial^N}{\partial x_1 \cdots \partial x_N} x_1^{\mu_1^s} \cdots x_N^{\mu_N^s}\big) f(x) d^N x|_{s=0} \\ = &\int_0^\infty r^{\nu-1} dr \int_{\hat{S}^\infty} f(x) \prod_{n=1}^\infty \sin^{\nu-n-1}\theta_n d\theta_n. \end{aligned}$$

Because $\theta_1, \theta_2, \dots$ are independent in this calculation. Summarizing these, we have ([3])

Theorem 2. *By the polar coordinate of $\hat{H} = \rho(H^\sharp)$, we have*

$$\rho^*(: d^\infty x :) = r^{\nu-1} dr d^\infty \omega, \quad d^\infty\omega = \prod_{n=1}^\infty \sin^{\nu-n-1}\theta_n d\theta_n. \tag{35}$$

Note 1. By the constraint (6), $d^\infty\omega$ can not defined on S^∞, the unit sphere of H.

Note 2. We need not integrity of ν in (35). Since $\nu(m)$ is a polynomial of m, if $\nu(m) = \zeta(G_m, 0)$, where G_m is the Green operator of $D + mI$, D is an elliptic operator, we ask; Can we give an interpretation of dimensional regualrization in renormalization theory via (35)?

Since $\int_{H^\sharp} \mathrm{e}^{-\pi\|x\|^2} : d^\infty x := 1$, by (30), we have

$$\int_0^\infty r^{\nu-1}\mathrm{e}^{-\pi r^2} dr \int_{\hat{S}^\infty} d^\infty\omega = 1.$$

Therefore we have

$$: \mathrm{vol}(\hat{S}^\infty) := \int_{\hat{S}^\infty} d^\infty\omega = \frac{2\pi^{\nu/2}}{\Gamma(\frac{\nu}{2})}. \tag{36}$$

Note. To apply same calculation to $(\mathbb{R}^n)^\perp \oplus \mathbb{R}e_\infty \subset H^\sharp$, we have

$$: d^\infty x := d^N x r^{\nu-N} dr : d^{\infty-N}\omega : .$$

This suggests existence de Rham type cohomology of degree $(\infty - N)$ of $H^\sharp \setminus \mathbb{R}^N$ and residue map

$$\mathrm{res} : H^{\infty-N}(H^\sharp \setminus \mathbb{R}^N, \mathbb{R}) \to H^0(\mathbb{R}^N, \mathbb{R}).$$

Definition and existence of the residue map are investigated in [4].

8 Regularized Cauchy kernel

Let H be a complex Hilbert space, and let $\mathbb{C}^N = \{\sum_{n=1}^N z_n e_n\} \subset H$. We also set

$$T_r^{\infty,k} = \{\sum_{n=1}^\infty z_n e_{n,k} \in W^{k,\sharp} | |z_n| = \mu_n^{d/2} r\}, \tag{37}$$

$$T_r^{N,k} = T_r^{\infty,k} \cap \mathbb{C}^N, \quad \mathrm{D}_r^{N,k} = \{\sum_{n=1}^N z_n e_{n,k} | |z_n| \le \mu_n^{d/2} r\} \tag{38}$$

If $k = -d/2$, $r = 1$, we denote T^∞ and D^∞, instead of $T?_1^{\infty,-d/2}$ and $\mathrm{D}_1^{\infty,-d/2}$.

By the map $w = z^a$, $\{z = \mathrm{e}^{\mathrm{i}\theta} | 0 \le \theta \le 2\pi\}$ is mapped to $\{w = \mathrm{e}^{\mathrm{i}\phi} | 0 \le \phi \le 2a\pi\}$, we have

$$\frac{(2\pi\mathrm{i})^{a-1}}{a} \int_{|z|=1} \frac{d(z^a)}{z^a} = (2\pi\mathrm{i})^a.$$

Here $\int_{|z|=1} dz$ means $\int_0^{2\pi} \mathrm{i}e^{\mathrm{i}\theta} d\theta$. Hence we have

$$\lim_{n\to\infty} \int_{T^n} \Big(\frac{(2\pi \mathrm{i})^{\mu_1^s - 1}}{\mu_1^s} \frac{d(z_1^{\mu_1^s})}{z_1^{\mu_1^s}} \cdots \frac{(2\pi \mathrm{i})^{\mu_n^s - 1}}{\mu_n^s} \frac{d(z_n^{\mu_n^s})}{z_n^{\mu_n^s}}\Big)|_{s=0} = (2\pi \mathrm{i})^{\nu}. \tag{39}$$

$$T^n = \{e^{\mathrm{i}\theta_1} | 0 \le \theta_1 \le 2\pi\} \times \cdots \times \{e^{\mathrm{i}\theta_n} | 0 \le \theta_n \le 2\pi\}.$$

Note that T^n is not a torus. But if ν is an integer, we can regard T^n to be a torus. Hence if ν is an integer, we can regard

$$T^{\infty} = \{e^{\mathrm{i}\theta_1} | 0 \le \theta_1 \le 2\pi\} \times \{e^{\mathrm{i}\theta_2} | 0 \le \theta_2 \le 2\pi\} \times \cdots .$$

Then we have

$$\int_{T^\infty} \frac{: d^\infty z :_{T^\infty}}{: \prod_{n=1}^\infty z_n :_G} = (2\pi \mathrm{i})^{\nu}, \tag{40}$$

$$: d^\infty z :_{T^\infty} = \prod_{n=1}^{\infty} \Big(\frac{(2\pi \mathrm{i})^{\mu_n^s - 1}}{\mu_n^s} d(z_n^{\mu_n^s})\Big)|_{s=0}. \tag{41}$$

Hence if a function $f(z)$ on $\mathrm{D}^\infty = \{\sum_n z_n e_n | |z_n| < \mu_n^{d/2}\}$ allows Taylor expansion

$$f(z) = \sum_{i_1, \dots, i_k} a_{i_1, \dots, i_k} z_1^{i_1} \cdots z_k^{i_k}, \quad |z_1| < \mu_1 \ d/2, \dots, |z_k| < \mu_k^{d/2},$$

we have

$$f(\zeta) = \frac{1}{(2\pi \mathrm{i})^\nu} \int_{T^\infty} f(z) \frac{: d^\infty z :_{T^\infty}}{: \prod_{n=1}^\infty (z_n - \zeta_n) :}. \tag{42}$$

Definition 4. *We say* $\frac{1}{(2\pi \mathrm{i})^\nu} \frac{: d^\infty z :_{T^\infty}}{: \prod_{n=1}^\infty (z_n - \zeta_n) :}$ *to be the regularized Cauchy kernel, provided* ν *is an integer.*

Note. If ν is not an integer, we can not expect the formula (42). Because $: \prod_n (z_n - \zeta_n) :$ is not a single valued function if ν is not an integer.

A function $f(z)$ on dense subset of $\mathcal{D}^\infty$ is said to be analytic if

$$\frac{\partial f(z)}{\partial \bar{z}_n} = 0, \ n = 1, 2, \dots ,$$

According to this definition, $: \prod_{n=1}^\infty z_n :$ is an analytic function. But it does not allow Taylor expansion.

If $|c| > |z|$, we have $(z+c)^a = c^a\big(1+\sum_{n=1}^{\infty}\frac{a(a-1)\cdots(a-n+1)}{n!}(\frac{z}{c})^n\big)$.
Hence we have

$$\begin{aligned}&\int_{|z|=\epsilon}(z+c)^a\frac{d(z^a)}{z^a}\\ =\ & a\mathrm{i}c^a\int_{|z|=\epsilon}\big(1+\sum_{n=1}^{\infty}\frac{a(a-1)\cdots(a-n+1)}{n!}\frac{\epsilon^n}{c^n}\mathrm{e}^{\mathrm{i}n\theta}\big)d\theta = 2a\pi\mathrm{i}c^a.\end{aligned}$$

Therefore we get

$$\frac{1}{(2\pi\mathrm{i})^{\nu}}\int_{T^{\infty}} :\prod_{n=1}^{\infty} z_n : \frac{:d^{\infty}z:_{T^{\infty}}}{:\prod_{n=1}^{\infty}(z_n-\zeta_n):} =: \prod_{n=1}^{\infty} c_n :, \quad |c_n| > \mu_n^{d/2}. \tag{43}$$

On the other hand, $(z+c)^a = z^a\big(1+\sum_{n=1}^{\infty}\frac{a(a-1)\cdots(a-n+1)}{n!}(\frac{c}{z})^n\big)$ if $|z| > |c|$. Hewnce we have

$$\begin{aligned}&-\int_{|z|=r}(z+c)^a\frac{d(z^a)}{z^a}\\ =\ & \int_0^{2\pi} ra\mathrm{i}\big(\mathrm{e}^{\mathrm{i}a\theta}+\sum_{n=1}^{\infty}\frac{a(a-1)\cdots(a-n+1)}{n!}(\frac{c}{r})^n\mathrm{e}^{\mathrm{i}(a-n)\theta}\big)d\theta\\ =\ & ra\mathrm{i}(\mathrm{e}^{2a\pi\mathrm{i}}-1)\big(1+\sum_{n=1}^{\infty}\frac{a(a-1)\cdots(a-n+1)}{n!}\frac{(\epsilon/r)^n}{\mathrm{i}(a-n)}\big)\\ =\ & 2\pi\mathrm{i}c + O(|a-1|), \quad a\to 1.\end{aligned}$$

Therefore $\frac{1}{(2\pi\mathrm{i})^{\nu}}\int_{T^{\infty}} :\prod_{n=1}^{\infty} z_n : \frac{:d^{\infty}z:_{T^{\infty}}}{:\prod_{n=1}^{\infty}(z_n-c_n):} = \prod_{n=1}^{\infty}$, if $|c_n| < \mu_n^{d/2}$. But since this infinite product vanishes, we obtain

$$\frac{1}{(2\pi\mathrm{i})^{\nu}}\int_{T^{\infty}} :\prod_{n=1}^{\infty} z_n : \frac{:dz^{\infty}z:_{T^{\infty}}}{:\prod_{n=1}^{\infty}(z_n-c_n):} = 0, \quad |c_n| < \mu_n^{d/2}. \tag{44}$$

Therefore, behavior of $:\prod_{n=1}^{\infty} z_n:$ with respect to the regularized Cauchy kernel is similar to the behavior of the principal part of a Laurent series with respect to the Cauchy kernel.

9 Fourier expansion of periodic functions on $H^{\sharp}$

Let H be a real Hilbert space. The free abelian group generated by $\mu_n^{d/2}e_n$; $n=1,2,\dots$, in H is denoted by $\mathbb{Z}^{\infty}$. Its closure in $H^{\sharp}$ is denoted by $\hat{\mathbb{Z}}^{\infty}$. As modules, we have

$$\hat{\mathbb{Z}}^{\infty} = \mathbb{Z}^{\infty} \oplus \mathbb{Z}e_{\infty}. \tag{45}$$

We set $\mathbb{T}^\infty = H/\mathbb{Z}^\infty$, $\hat{\mathbb{T}}^\infty = H^\sharp/\hat{\mathbb{Z}}^\infty$. Then we get

$$\hat{\mathbb{T}}^\infty = \mathbb{T}^\infty \times S^1, \quad S^1 = \mathbb{R}e_\infty/\mathbb{Z}e_\infty. \tag{46}$$

We denote p_1 and p_∞, the projections from $\hat{\mathbb{T}}^\infty$ to $\mathbb{T}^\infty$ and $S^1 = \mathbb{R}e_\infty/\mathbb{Z}e_\infty$, respectively. Then denoting $C_b^1(\hat{\mathbb{T}}^\infty)$, $C_b^1(\mathbb{T}^\infty)$ and $C^1(S^1)$, the spaces of Fréchet differentiable C^1-class functions on $\hat{\mathbb{T}}^\infty$, $\mathbb{T}^\infty$ and on $S^1 = \mathbb{R}e_\infty/\mathbb{Z}e_\infty$, respectively, $p_1^*(C_b^1(\mathbb{T}^\infty)) \times p_\infty^*(C^1(S^1))$ is dense in $C_b^1(\hat{\mathbb{T}}^\infty)$ by the C^1-topology.

By the definition of $\mathbb{Z}^\infty$, functions in $C_b^1(\mathbb{T}^\infty)$ allows Fourier expansion by the finite products of $s_{n_m} = \sin(2n_m\mu_n^{-d/2}x_n)$ and $c_{n_m} = \cos(2n_m\mu_n^{-d/2})$. While, via the map $\rho : H^\natural \cong \hat{H}$, functions in $C^1(S^1)$ is expanssed by using infinite products of s_{n_m} and c_{n_m}. In the rest, we set

$$e_I(x) = \prod_{n=1}^{\infty} *_{m_n}, \quad I = (m_{1,*}, m_{2.*}, \dots),$$

where $*$ is either of s or c. As remarked in §3, $\lim_{n\to\infty} = m_\infty$ should exists and except finite numbers, $*$'s are s (or c). Then $f \in C_b^1(\hat{\mathbb{T}}^\infty)$ is expanssed as

$$f(x) = \sum_I c_I e_I(x), \quad \sum_I |c_I|^2 < \infty.$$

We can take

$$\mathcal{D}_{0,\mu^{d/2}} = \{\sum_{n=1}^{\infty} x_n e_n | 0 \le x_n \le \mu_n^{d/2}\}, \tag{47}$$

$0 = (0, 0, \dots)$, $\mu^{d/2} = (\mu_1^{d/2}, \mu_2^{d/2}, \dots)$, as a fundamental domain of $\hat{\mathbb{Z}}^\infty$. Then by Corollary of Theorem 1, we have

$$\int_{\mathcal{D}_{0,\mu^{d/2}}} e_I(x)e_J(x) : d^\infty x := 0, \quad I \neq I, \tag{48}$$

$$\int_{\mathcal{D}_{0,\mu^{d/2}}} e_I(x)^2 : d^\infty x := \epsilon_I, \tag{49}$$

$$\epsilon_I = \begin{cases} \frac{1}{2^k}(\det G)^{d/2}, & *_n \neq 1 \text{ except } n \in \{n_1, \dots, n_k\}, \\ \frac{1}{2^{\nu-k}}(\det G)^{d/2}, & *_n = 1 \text{ except } n \in \{n_1, \dots, n_k\}. \end{cases} \tag{50}$$

Hence we have

Theorem 3. *Let f be a function in $C_b^1(\hat{\mathbb{T}}^\infty)$, then*

$$f(x) = \sum_I \frac{1}{\epsilon_I} \int_{\mathcal{D}_{0,\mu^{d/2}}} f(x)e_I(x) : d^\infty x : e_I(x). \tag{51}$$

Note We denote $\int_{\hat{\mathbb{T}^\infty}} f(x) : d^\infty x :$ instead of $\int_{\mathcal{D}_{0,\mu^{d/2}}} f(x) : d^\infty x :$. Using this notation, we have

$$\int_{\hat{\mathbb{T}}^\infty} f(x)^2 : d^\infty x := \sum_I \frac{1}{\epsilon_I} \big(\int_{\hat{\mathbb{T}}^\infty} f(x) e_I(x) : d^\infty x : \big)^2,$$

by (51). Therefore the inner product $(f,g) = \int_{\hat{\mathbb{T}}^\infty} f(x)g(x) : d^\infty x :$ is positive definite. Hence we can define the Hilbert space $L^2(\hat{\mathbb{T}}^\infty)$, which is the completion of $C_b^1(\hat{\mathbb{T}}^\infty)$ by the norm $\|f\|^2 = \int_{\hat{\mathbb{T}}^\infty} |f(x)|^2 : d^\infty x :$. Its complete ortho-normal basis is given by $\{\frac{e_I(x)}{\sqrt{\epsilon_I}} | I = (m_{1,*}, m_{2,*}, \dots)\}$.

10 Periodic boundary value problem of regularized Lapalcian

We have defined regularized Laplacian $: \triangle :$ in Introduction. But the deinition of regularized Laplacian stated in Introduction is a little restricitive. In this Section, we define regularized Laplacian by

$$: \triangle : f = \triangle(s) f_s|_{s=0}, \quad f_s(x)|_{s=0} = f(x), \tag{52}$$

$$\triangle(s) = \sum_{n=1}^\infty \mu_n^s \frac{\partial^2}{\partial x_n^2}. \tag{53}$$

We treat its periodic boundary value problem

$$f(x)|_{x_n=0} = f(x)|_{x_n=\mu_n^{d/2}}, \quad \frac{\partial f}{\partial x_n}|_{x_n=0} = \frac{\partial f}{\partial x_n}|_{x_n=\mu_n^{d/2}}, \tag{54}$$

$n = 1, 2, \dots$, as an application of Fourier expansion of periodic functions on $H^\sharp$ with the period $\hat{\mathbb{Z}}^\infty$ in §9.

If an eigenfunction $f_s(x)$ of $\triangle(s)$ with respect to the boundary condition (54) takes the form $\prod_{n=1}^\infty f_n(x_n)$, then we have

$$f_n(x_n) = A_n \sin(2m_n \pi \mu_n^{s-d/2} x_n) + B_n \cos(2m_n \pi \mu_n^{s-d/2} x_n).$$

As remarked in §3, if $f(x)$ does not vanish on $H^\sharp$, then $\lim_{n\to\infty} m_n = m_\infty$ should be exist. Assume $A_n = 0$, $B_n = 1$, or $A_n = 1$, $B_n = 0$, except finite factors, $f_s(x)$ is an infinite product of

$$\sin(2m_\infty \pi \mu_n^{s-d/2} x_n), \text{ or } \cos(2m_\infty \pi \mu_n^{s-d/2} x_n).$$

Then we have

$$\triangle(s)f_s(x) = -(m_\infty^2\zeta(G,2s-d) + \sum_{n=1}^N (m_n^2 - m_\infty^2)\mu_n^{2s-d})f_s(x).$$

By the definition of $f_s(x)$, $f(x) = f_s(x)|_{s=0}$ is an infinite product of

$$\sin(2m_\infty\pi\mu_n^{-d/2}x_n), \text{ or } \cos(2m_\infty\pi\mu_n^{-d/2}x_n),$$

except finite factors. So they form complete orthogonal basis of $L^2(\hat{\mathbb{T}}^\infty)$, and if $\zeta(G,s)$ is holomorphic at $s=-d$, then

$$:\triangle: f(x) = -(m_\infty^2\zeta(G,-d) + \sum_{n=1}^N (m_n^2 - m_\infty^2)\mu_n^{-d})f(x). \quad (55)$$

Hence we have

Theorem 4 *If $\zeta(G,s)$ is holomorphic at $s=-d$, then the eigenvalues of $:\triangle:$ with respect to the boundary condition* (54) *are*

$$\{-m_\infty^2\zeta(G,-d) + \sum_{n=1}^N (m_n^2 - m_\infty^2)\mu_n^{-d} | m_\infty, m_n \in \mathbb{N}\cup\{0\}$$

. Eigenfunctions belonging to this eigenvalues form complete orthogonal basis of $L^2(\hat{\mathbb{T}}^\infty)$.

Note. If we consider same boundary value problem on H, eigenfunctions belong to $\zeta(G,-d)$ vanishes. Hence eigenvalues and eigenfunctions all come from finite dimensional Laplacian.

Introducing Clifford algebra with $(\infty - p)$-spinors, we can define Dirac opertor $\not{D}$ and its regularization $:\not{D}:$ on $H^\sharp$. Then, if $\zeta(G,s)$ is holomorphic at $s=-d/2$, eigenvalues of $:\not{D}:$ with respect to the boundary condition $f|_{x_n=0} = f|_{x_n=\mu_n^{d/2}}$, $n=1,2,\dots$ are

$$\{m_\infty\zeta(G,-d/2) + \sum_{n=1}^N (m_n - m_\infty)\mu_n^{-d/2} | m_\infty, m_n \in \mathbb{Z}\}.$$

Note that $\zeta(G,s)$ is holomorphic at $s=-d/2$, if G is the Green operator of an elliptic operator on an odd dimensional compact manifold.

$\triangle$ does not allow polar coordinate expression. But $:\triangle:$ allows the following polar coordiante expression:

$$\begin{aligned} :\triangle: &= \frac{\partial^2}{\partial r^2} + \frac{\nu-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda[\nu], \\ \Lambda[\nu] &= \sum_{n=1}^\infty \frac{1}{\sin^2\theta_1\cdots\sin^2\theta_{n-1}}\left(\frac{\partial^2}{\partial\theta_n^2} + (\nu-n-1)\frac{\cos\theta_n}{\sin\theta_n}\frac{\partial}{\partial\theta_n}\right). \end{aligned}$$

We can regard $\Lambda[\nu]$ to be the (regularized) spherical Laplacian. If ν is an integer, eigenvalues of $\Lambda[\nu]$ on $\hat{S}^\infty$ are

$$l_n(l_n+\nu-n-2),\ l_n\in\mathbb{N},\quad l_1\geq L_2\geq\ldots\geq 0.$$

Eigenfunctions belonging to these eigenvalues are finite product of Gegenbauer polyomials and infinite product of

$$\int_0^{\theta_n}\sin^{n+1-\nu-2l_\infty}\theta_n d\theta_n,\quad \lim_{n\to\infty}l_n=l_\infty.$$

This latter eigenfunctions do not have finite dimensional analogy. But they vanish on S^∞. Hence on S^∞, all eigenvalues and eigenfunctions of $\Lambda[\nu]$ come from finite dimensional spherical Laplacian ([1],[17]).

11 Remarks on $(\infty-p)$-forms

We can define $:\prod_{n\notin\{i_1,\dots,i_p\}}x_n:$ by $\prod_{n\notin\{i_1,\dots,i_p\}}x_n^{\mu_n^s}|_{s=0}$ and we have

$$:\prod_{n\notin\{i_1,\dots,i_p\}}x_n:=\frac{\partial^p}{\partial x_{i_1}\cdots\partial x_{i_p}}:\prod_{n=1}^\infty x_n:=\frac{1}{x_{i_1}\cdots x_{i_p}}:\prod_{n=1}^\infty x_n:. \tag{56}$$

Similarly, we can define $:d^{\infty-\{i_1,\dots,i_p\}}x:$ as the regularized volume form on $(\mathbb{R}^p_{i_1,\dots,i_p})^\perp\oplus\mathbb{R}e_\infty\subset H^\sharp$. Here $\mathbb{R}^p_{i_1,\dots,i_p}$ means the subspace of H spanned by $e_{i_1},\dots,e_{i_p}$. If we regard $:d^{\infty-\{i_1,\dots,i_p\}}x:$ to be the $(\infty-p)$-form $\Lambda_{n\notin\{i_1,\dots,i_p\}}dx_n$ on $H^\sharp$, the commutation rule should be

$$\begin{aligned}&dx_{i_1}\wedge\ldots\wedge dx_{i_p}\wedge\Lambda_{n\notin\{i_1,\dots,i_p\}}dx\\ =\ &(-1)^{p(\nu-p)}\Lambda_{n\notin\{i_1,\dots,i_p\}}dx\wedge dx_{i_1}\wedge\ldots\wedge dx_{i_p}.\end{aligned} \tag{57}$$

This rule is well defined if ν is an integer. But not defined if ν is not an integer, because $(-1)^{p(\nu-p)}$ is not a real number, in general.

Since a p-form on U, an open set of W^k, is a smooth map to $\Lambda^pW\ -k$, we can define an $(\infty-p)$-form to be a smooth map $f:U\to\Lambda^pW^k$, by the Sobolev duality. We regard f to be an alternative function form U to $\overbrace{W^k\otimes\cdots\otimes W^k}^{p}$.

Definition 5. *Let $\hat{d}f$ be the Fréchet differential of f. Then we define exterior differential df of f by*

$$df(x;x_1,\dots,x_{p-1})=(-1)^{p-1}\mathrm{tr}\big(\hat{d}f(x;x_1,\dots,x_{p-1},x)\big). \tag{58}$$

Here we regard $\hat{d}f(x;x_1,\dots,x_{p-1},x)$ to be a map from U to the space of linear operators of W^k, with parameters $x_1,\dots,x_{p-1}$.

We can not expect $\hat{d}f(x;,\ldots,x)$ to be a map to the ideal of trace class operators. For example, $\omega=\sum_{n=1}^{\infty}(-1)^{n-1}x_n d^{\infty-\{n\}}x$ is not exteior differentiable. In fact, an $(\infty-p)$-from is (global) exact if it is exterior differentiable. As a consequence, d is not nilpotent on the sapce of $(\infty-p)$-froms ([1]). Therefore, if we use above definition of exterior sifferential, we can not obtain non-trivial de Rham theory having $(\infty-p)$-th cohomology groups.

On the other hand, according to our previous regularization procedure, regualrized exterior differential should be defined by

$$:d:f(x;x_1,\ldots,x_{p-1})=(-1)^{p-1}\mathrm{tr}(G^s\hat{d}f(x;x_1,\ldots,x_{p-1},x)|_{s=0}. \tag{59}$$

For example, we have

$$:d:\omega=\big(\sum_{n=1}^{\infty}\mu_n^s\big)d^{\infty}x|_{s=0}=\nu d^{\infty}x.$$

Since $G^k:W^{-k}\cong W^k$, we may consider G^k (and $\Lambda^pG^k:\Lambda^pW^k\cong\Lambda^pW^{-k}$) to be the Hodge $*$-operator. Then we have

$$*^{-1}:d:*df=:\triangle:f. \tag{60}$$

At this stage, we do not know porperties of $:d:$ such as validities of Poincaré Lemma and nilpotency. So we can not construct de Rham theory by using $:d:$.

But on $\hat{\mathbb{T}}^{\infty}$, we can construct de Rham type cohomology with $(\infty-p)$-th degree eleemnts as follows: Differential forms $dx_{i_1}\wedge\ldots\wedge dx_{i_p}$ and $:d^{\infty-\{i_1,\cdots,i_p\}}x:$ (regarded to be $\Lambda^{\infty-\{i_1,\cdots,i_p\}}x$) on $\mathcal{D}_{0,\mu^{d/2}}$ induce differential forms on $\hat{\mathbb{T}}^{\infty}$. If ν is an integer, by the commutation rule (57), we have an algebra $H^{*,*}(\hat{\mathbb{T}}^{\infty},\mathbb{R})$. The subalgebra consisted by finite degree elements of this algebra is the cohomology algebra $H^{*}(\hat{\mathbb{T}}^{\infty},\mathbb{R})$ of $\hat{\mathbb{T}}^{\infty}$. Moreover, this algebra has the Poincaré duality by the pairing

$$(dx_{i_1}\wedge\ldots\wedge dx_{i_p},\Lambda^{\infty-\{i_1,\cdots,i_p\}}x)=\int_{\mathcal{D}_{0,\mu^{d/2}}}:d^{\infty}x:(=\det G)^{d/2}). \tag{61}$$

$H^{*,*}(\hat{\mathbb{T}}^{\infty},\mathbb{R})$ seems not a toplogical invariant. Its multiplicative structure depends on mod.2 class of ν.

If ν is not an integer, we can not define $H^{*,*}(\hat{\mathbb{T}}^{\infty},\mathbb{R})$. But to define commutation relation of a p-form ϕ^p and an $(\infty-q)$-form $\psi^{\infty-q}$ by

$$\phi^p\wedge\psi^{\infty-q} = \mathrm{e}^{p(\nu-q)\pi\mathrm{i}}\psi^{\infty-q}\wedge\phi^p, \tag{62}$$
$$\psi^{\infty-q}\wedge\phi^p = \mathrm{e}^{-p(\nu-q)\pi\mathrm{i}}\phi^p\wedge\psi^{\infty-q}, \tag{63}$$

we can define $H^{*,*}(\hat{\mathbb{T}}^\infty, \mathbb{C})$. Note that these commutation relations are closely related to that of noncommutative torus.

Note. $H^{*,*}(\hat{\mathbb{T}}^\infty, \mathbb{C})$ is isomorphic to the Grassmann algebra with $(\infty - p)$-th degree elements over $\mathbb{C}$. By the commutation relations (62), (63), such algebras are classified by the mod.2 classes of ν. We also note there is an alternative selection of the commutation relation, *e.g*;

$$\phi^p \wedge \psi^{\infty-q} = \mathrm{e}^{-p(\nu-q)\pi \mathrm{i}} \psi^{\infty-q} \wedge \phi^p.$$

This selection of the commutation relation corresponds to the change ν to $2-\nu$. Therefore we only need to consider commutaiton relations (62), (63).

We can also define $H^{*,*}(\hat{S}^\infty, \mathbb{R}) = H^0(\hat{S}^\infty, \mathbb{R}) \oplus H^\infty(\hat{S}^\infty, \mathbb{R})$ without any assumption on ν. The generator of $H^\infty(\hat{S}^\infty, \mathbb{R})$ is $: d^\infty\omega :$ and the pairing of $c \in H^0(\hat{S}^\infty, \mathbb{R})$ and $\omega \in H^\infty, \mathbb{R})$ is defined by

$$(c, \omega) = \int_{\hat{S}^\infty} ca : d^\infty\omega :, \quad \omega = a : d^\infty\omega : .$$

These examples suggest possibility of existence of de Rham type cohomology $H^{*,*}(M, \mathbb{C})$ of a Hilbert manifold M such that

$$H^{*,*}(M, \mathbb{C}) = H^*(M, \mathbb{C}) \oplus *(H^*(M, \mathbb{C})),$$

having Poincaré duality, where $H^*(M, \mathbb{C})$ is the complex coefficient de Rham cohomology of M and $*$ is the Hodge operator, if we can define weak topology of M and M is compact by the weak topology.

Precisely, if M is compact by the weak topology, then it seems M has a regularized volume $: dV :$ and $\int_M : dV :$ is finite. Similar to finite dimensional case, Poincaré duality is given by using $: dV :$. Since regularized volume fors on $\hat{S}^\infty$ and $\hat{\mathbb{T}}^\infty$ have singularities, $: dV :$ may have singularites.

References

[1] Asada, A.: Regularized Calculus: An application of the zeta regularization to infinite dimensional geometry and analysis, Int. J. Geom. Meth. Mod. Phys., 1(2004), 107-157.

[2] Asada, A.: Zeta-regularization and calculus on infinite-dimensional spaces, AIP Conf. Proc. 729(2004), 71-83.

[3] Asada, A.: Regularized volume form of the Hilbert space with the determinat bundle, Diff. Geom. and Its Appl., 397-409, matfyzpress, Prague 2004.

[4] Asada, A.: Fractional calculus and regularized residue of infinite dimensional space, Math. Meth. in Engineering, 3-11, Springer, 2007.

[5] Asada, A.: Regularized infinite product, regularized determinant and determinant bundel, preprint.

[6] Asada, A.: Zeta regularized integral and Fourier expansion of functions on an infinite dimensional torus, submitted.

[7] Booss-Bavnbek, B. Wojciechowski, K.: Elliptic Boundary Problems for Dirac Operators, Boston, 1993.

[8] Cardona, A. Ducourtioux, C. Paycha, S.: From tracial anomaly to anomalies in quantum field theory, Commun. Math. Phys., 242(2002), 31-65.

[9] Connes, A.: Noncommutaitve Geometry, Academic, 1994.

[10] Cuntz, J.: Cyclic Theory, Bivriant K-theory and the Chern-Connes Charcter, Cyclic Cohomology in Non-Commutative Geometry, EMS 121, 1-71, Springer, 2002.

[11] Gilkey, P.: The Geometry of Spherical Space Groups, World Sci. 1989.

[12] Léandre, R.: Theory of distributions in the space of Connes-Hida an Feynmann path integral on a manifold, Inf. Dim. Anal. Quant. Prob. and Rel. Topics, 6(2003), 505-517.

[13] Léandre, R.: Stochastic algebraic de Rham complexes, Acta Appl. Math., 79(2003), 217-247.

[14] Okikiolu, K.: The Campbell-Haussdorff theorem for elliptic operators and a related trace formula, The multiplicative anomaly for determinant of elliptic operators, Duke Math. J.,79(1995), 682-722, 723-750.

[15] Paycha, S.: Renormalized traces as a looking glass into infinite dimensional geometry, Inf. Dim. Anal. Qunat. Prob. and rel. Topics, 4(2001), 221-266.

[16] Simon, B.: Trace Ideals and Their Applications, Cambridge, 1979.

[17] Tanabe, N.: Regularized Hilbert space Laplacian and its spherical harmonics, Rev. Bull. Calcutta Math. Soc., 11(2003), 27-38.

[18] Wojciechowski, K.: The ζ-determinant and the additivity of the η-invariant on the smooth selfadjoint Grassmannian, Commun. Math. Phys.,201(1999), 423-444.