Euler-Lagrange EDS and Noether's Theorem

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1 Introduction

Classical mechanics:

- $L \in C^{\infty}(\mathrm{T}M^n)$ Lagrange function; locally, $L = L(q^i, \dot{q}^i)$
  $\Rightarrow$ Funtional $\mathcal{L}$ on the space of paths $\{l = q(t)\}$ in M :

$$\mathcal{L}(l) = \int_0^1 L(q(t), \dot{q}(t))dt.$$

By calculus of variation, obtain the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0.$$

Also, we have

- Symmetry of $L \Rightarrow$ Conservation law (1-st integrals)[Noether].

A similar arguement for the Legendre submanifolds in a contact manifold M of dimension $2n+1$.

- Lagrangian n-form Λ
  $\Rightarrow$ Functional $\mathcal{L}$ on the space of Legendrian submanifolds
  $\Rightarrow$ Euler-Lagrange differential system (Monge-Ampere differential system)

$$\mathcal{E}_\Lambda = \{\theta, d\theta, \Psi\}, \qquad \Psi : n\text{-form.}$$

What I will talk is a part of the discussion of **Exterior Differential Systems and Euler-Lagrange Prtial Differential Equations** (by R.Bryant, P.Griffiths and D. Grossman). University of Chicago Press,2003.(Book review Bull. of A.M.S. 2005 Vol.42)

2 Lagrangian forms on a contact manifold

Def 1 . Let M be a *conatct manifold*, $\dim M = 2n + 1$, θ contact 1-form.

(a) I : contact line bundle generate by θ, $I = \langle\theta\rangle$

(b) $\mathcal{I}$: ideal generated by $\{\theta, d\theta\}$ in exterior algebra $\Omega(M)$.

(c) A *Legendre submanifold* is an n-dimensional integral manifold of $\mathcal{I}$, i.e. $\theta|_N = 0$.

Examples

- $J^1(\mathbb{R}^n, \mathbb{R})$: 1-jets of funtions, $2n + 1$ dim.
- $G_n(\mathrm{T}\mathbb{R}^{n+1})$: set of n-dim oriented subspaces of tangent spaces. ($\cong$ sphere bundle over $\mathbb{R}^{n+1}$).

Lagrangian n-form

- (M, I) : contact manifold.
- Λ : n-form.
- N : Legendre submanifold.

Want consider the integral(functional)

$$\mathcal{F}_\Lambda(N) = \int_N \Lambda.$$

(1) If $\Lambda \in \mathcal{I}$, $\mathcal{F}_\Lambda(N) = 0$, $(N : \textit{Legendrian})$.
(2) If $\Lambda - \Lambda' = d\phi$ and we fix the ∂N ,

$$\int_N \Lambda - \int_N \Lambda' = \int_{\partial N} \phi.$$

Thus, better to consider $[\Lambda]$ as a class in $\Omega^n(M)$ modulo $\mathcal{I} + d\Omega^{n-1}$.

To see Λ is closed in the differential complex $\Omega(M)/\mathcal{I}$, we need

Lemma 1 .V symplectic vector space, Θ non-deg.bi-lin.form.

- $\bigwedge^{n-k}(V^*) \xrightarrow{\Theta^k\wedge} \bigwedge^{n+k}(V^*)$ is an isomorphism, $(0 \leq k \leq n)$.
- $\Theta^{k+1}\wedge : \bigwedge^{n-k}(V^*) \longrightarrow \bigwedge^{n+k+2}(V^*)$ gives a decomposition

$$\bigwedge^{n-k}(V^*) \cong \ker(\Theta^{k+1}\wedge) \oplus (\Theta \wedge \bigwedge^{n-k-2}(V^*)).$$

In particular, $\bigwedge^{n+1}(V^*)$ is in image$(\Theta\wedge)$.

The elements in $\ker(\Theta^{k+1}\wedge)$ are called *primitive*.

Apply the lemma for $\Omega(M)/\langle\theta\rangle$. Then, $d\Lambda \in \Omega^{n+1}(M)/\langle\theta\rangle$ is in the image of $(d\theta\wedge)$ i.e. $d\Lambda = 0$ in $\Omega(M)/\mathcal{I}$. Our Lagrangian is an element $\boxed{[\Lambda] \in H^n(\Omega(M)/\mathcal{I})}$.

Poincaré-Cartan form

From the short exact sequence of complexes

$$0 \to \mathcal{I} \to \Omega(M) \to \Omega(M)/\mathcal{I} \to 0,$$

we have

$$\cdots \to H^n_{dR}(M) \to H^n(\Omega(M)/\mathcal{I}) \xrightarrow{\delta} H^{n+1}(\mathcal{I}) \to \cdots$$

The *P-C form* is the $(n+1)$-form Π representing $\delta[\Lambda]$, which is *uniquely* defined on the condition $\theta \wedge \Pi = 0$ (i.e. $\Pi = 0, \mathrm{mod}\ \langle\theta\rangle$).

By $d\Lambda = \theta \wedge \alpha + d\theta \wedge \beta = \theta \wedge (\alpha + d\beta) + d(\theta \wedge \beta)$ the existence of Π s.t. $\theta \wedge \Pi = 0$ is clear. Uniqueness of Π is proved by using the Lemma.
Thus $\Pi = \theta \wedge (\alpha + d\beta)$.
Also, since $d\Lambda \in \Omega^{n+1}$, there exists β' such that $d\theta \wedge \beta' = \Lambda$, mod $(I = \langle\theta\rangle)$ (Lemma). This β satisfies $d(\Lambda - \theta \wedge \beta') = 0$, mod $\{\theta\}$. By the uniqueness, $\Pi = d(\Lambda - \theta \wedge \beta')$. Thus, we have $\boxed{\Pi = \theta \wedge (\alpha + d\beta) = d(\Lambda - \theta \wedge \beta')}$
The last β' is uniquely determined element (mod$I$) such that $\Lambda - \theta \wedge \beta'$ is closed (mod$I$).

3 Variation of a Legendre submanifold

(a) (M, I); a contact manifold, $I = \langle\theta\rangle$.

(b) $(N, \partial N)$; C^∞ compact n-dim. $F : N \times [0,1] \to M$, 1-para. family of Legendre submfds: $F^*\theta|_{N\times\{t\}} = 0$, $F|_{\partial N\times[0,1]} = F|_{\partial N\times\{0\}}$.

(c) $[\Lambda] \in H^n(\Omega(M)/\mathcal{I})$, $\quad \Pi = d(\Lambda - \theta\wedge\beta)$: Poincaré-Cartan form.

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_\Lambda(F(N\times\{t\})) &= \frac{d}{dt}\int_{F(N\times\{t\})}\Lambda = \frac{d}{dt}\int_{N\times\{t\}} F^*\Lambda \\
&= \frac{d}{dt}\int_{N\times\{t\}} F^*(\Lambda-\theta\wedge\beta) = \int_{N\times\{t\}} \mathcal{L}_{\frac{\partial}{\partial t}} F^*(\Lambda-\theta\wedge\beta) \\
&= \int_{N\times\{t\}} \iota_{\frac{\partial}{\partial t}}(dF^*(\Lambda-\theta\wedge\beta)) + \int_{N\times\{t\}} d(\iota_{\frac{\partial}{\partial t}} F^*(\Lambda-\theta\wedge\beta))) \\
&= \int_{N\times\{t\}} \iota_{\frac{\partial}{\partial t}}(F^* d(\Lambda-\theta\wedge\beta)) + \int_{\partial N\times\{t\}} \iota_{\frac{\partial}{\partial t}} F^*(\Lambda-\theta\wedge\beta) \\
&= \int_{N\times\{t\}} F^*(\iota_{F_*\frac{\partial}{\partial t}}\Pi) + \int_{\partial N\times\{t\}} F^*(\iota_{F_*\frac{\partial}{\partial t}}(\Lambda-\theta\wedge\beta)) \\
&= \int_{F(N\times\{t\})} \iota_{F_*(\frac{\partial}{\partial t})}\Pi + 0.
\end{aligned}$$

If $F(N\times\{0\})$ is extremal(stationary)

$$\begin{aligned} 0 =& \frac{d}{dt}\int_{N\times\{t\}} F^*\Lambda|_{t=0} = \int_{F(N\times\{0\})} \iota_{F_*(\frac{\partial}{\partial t})}\Pi \\ =& \int_{F(N\times\{0\})} \iota_{F_*(\frac{\partial}{\partial t})}(\theta\wedge\Psi) = \int_{F(N\times\{0\})} \theta(F_*\frac{\partial}{\partial t})\Psi \end{aligned}$$

Locally, $\theta = dy - \sum_{i=1}^n z_i dx^i$ and $F(N\times\{0\})$ is $\{x,0,0\}$. For arbitrary function $g(x)$, $F(N\times\{t\}) = (x, tg(x), t\frac{\partial g}{\partial x})$ is Legendre and $\theta(F_*\frac{\partial}{\partial t}) = g(x)$.

Thus $\boxed{F(N\times\{0\}) \text{ is extremal Legendre mfd} \Leftrightarrow \Psi = 0 \text{ on } F(N\times\{0\})}$.

$N = F(N\times\{0\})$ is an integral manifold of E.D.S., called *Euler-Lagrange differential system*

$$\{\theta, d\theta, \Psi\}_{\text{alg}}.$$

Def 2 . Generally, a *Monge-Ampére differential system* on (M, I) is $\boxed{\mathcal{E} = \{\theta, d\theta, \Psi\}_{\text{alg}}}$ where Ψ is an n-form. $(n \geq 2)$.

E-L differential system is a Monge-Ampére sytem. In E-L system, since $\Pi = \theta\wedge\Psi$ is closed $d\theta\wedge\Psi = 0$ (mod θ), i.e. Ψ is primitive (mod θ).

Classical M-A equation is a 2nd order diff. equation for $z(x, y)$
On $J^2(\mathbb{R}^2, \mathbb{R})$, with coord. (x, y, z, p, q, r, s, t)

$$Ar + 2Bs + Ct + D + E(rt - s^2) = 0, \quad A = A(x, y, z, p, q), \ldots.$$

This is equiv. to a EDS on $J^1(\mathbb{R}^2, \mathbb{R})$

$\{\theta = dz - pdx - qdy, -d\theta = dp \wedge dx + dq \wedge dy, \Psi\} \qquad (n = 2)$
$\Psi = Adp \wedge dy + B(dq \wedge dy - dp \wedge dx) - Cdq \wedge dx + Ddx \wedge dy + Edp \wedge dq.$

Inverse problem

When M-A system is an E-L system?

Theorem(B.G.G.) Let $\mathcal{E} = \{\theta, d\theta, \Psi\}$ be a M-A system on $M^{2n+1}, n \geq 2$. Assume Ψ is primitive. Then $\mathcal{E}$ is locally equivalent to E-L system iff

$$d(\theta \wedge \Psi) = \phi \wedge (\theta \wedge \Psi)$$

where ϕ satisfies $d\phi = 0 \pmod{\mathcal{I}}$.
(c.f. Theory of variational bi-complex)

4 Conservation Laws and Noether's Theorem

Noether's Theorem in classical mechanics

(equivariant moment map $\Rightarrow$ constants of motion) [Hamiltonian mechanics]

Lagrangian formulation: vector field preserving Lagrangian function $L : TM \to \mathbb{R} \Rightarrow$ 1-st integrals of Euler-Lagrange eq.

X a vector field on M, $\tilde{X}$: prolongation to TM. Assume $\tilde{X}$ preserves the Lagrangian L. There is a *canonical* $(1,1)$-*tensor* Φ on TM, written locally as $\frac{\partial}{\partial \dot{q}^i} \otimes dq^i$. Then

$$\Phi(dL, \tilde{X})$$

is a first integral. Locally, $\Phi(dL, \tilde{X}) = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}^i} X^i \qquad \left(X = \sum X^i \frac{\partial}{\partial q^i} \right)$.
Legendre transf. $TM \to T^*M$ is given by

$$(q^i, \dot{q}^i) \to \Phi(dL, \cdot) = \frac{\partial L}{\partial \dot{q}^i} dq^i.$$

In the present case$(M, \mathcal{E}_\Lambda = \{\theta, d\theta, \Psi\})$, symmetries are the the following vector fields:

$$\mathcal{G}_{[\Lambda]} = \{v \in \mathcal{V}(M) | \mathcal{L}_v I \subset I, \mathcal{L}_v[\Lambda] = 0\}.$$

$$\mathcal{G}_\Pi = \{v \in \mathcal{V}(M) | \mathcal{L}_v \Pi = 0\}.$$

Rem 1 Can be shown $\mathcal{G}_{[\Lambda]} \subset \mathcal{G}_\Pi$ and $\mathcal{L}_v I \subset I$ for $v \in \mathcal{G}_\Pi$, if Λ is *nondegen.* i.e. $\Pi = \theta \wedge \Psi$ is not divisible by a 1-form other than θ.

1-st integral is replaced by $(n-1)$-form

Def 3 The space of *conservation laws* for EDS $\mathcal{E}_\Lambda$ is $H^{n-1}(\Omega(M)/\mathcal{E}_\Lambda)$.

A conservation law is an $(n-1)$-form which is closed on the integral manifolds. As before, we have an exact sequence

$$\cdots \to H^{n-1}_{dR}(M) \xrightarrow{i} H^{n-1}(\Omega(M)/\mathcal{E}_\Lambda) \xrightarrow{\delta} H^n(\mathcal{E}_\Lambda) \to \cdots$$

$\mathrm{Im}\delta \cong H^{n-1}(\Omega(M)/\mathcal{E}_\Lambda)/\mathrm{Im}(i)$ is called *space of proper conservation laws.*

The following is a version of Noether's Theorem.

Theorem 2 . (B.G.G.) Let $(M, \mathcal{E}_\Lambda)$ be an E-L system, where Λ is non-degenerate. Then there is an *isomorphism*

$$\eta : \mathcal{G}_\Pi \to H^n(\mathcal{E}_\Lambda)$$

which is given by $v \to \eta(v) = [\iota_v \Pi]$(interior product) and induces

$$\eta(\mathcal{G}_{[\Lambda]}) = \mathrm{Im}\delta \ (= \text{space of } \textit{proper conservation laws}).$$

<u>$* * * * * * * * * * *$</u>

For $v \in \mathcal{G}_{[\Lambda]}$, we have a conservation law $[\phi_v]$

$$\cdots \to H^{n-1}_{dR}(M) \xrightarrow{i} H^{n-1}(\Omega(M)/\mathcal{E}_\Lambda) \xrightarrow{\delta} H^n(\mathcal{E}_\Lambda) \to \cdots$$

$$\begin{array}{ccc} [\phi_v] & \to & \eta(v) \\ \text{consv.law} & & \text{prop.consv.law} \end{array}$$

A local formula for representaive ϕ_v is described using the data:

- $\Pi = d(\Lambda - \theta \wedge \beta)$, (P-C from)
- $v \in \mathcal{G}_{[\Lambda]}$ i.e. $\mathcal{L}_v \Lambda = d\gamma$, for some $(n-1)$-form γ (mod $\mathcal{I}$).

Then $\boxed{\phi_v = -\iota_v \Lambda + \theta(v)\beta + \gamma}$. satisfies $\delta[\phi_v] = [d\phi_v] = \iota_v \eta$.

proof (1) $d\phi_v$ is calculated to be $\iota_v \Pi$, (mod $\mathcal{I}$). (a direct computaion).
(2) By $\mathcal{L}_v \Pi = 0$, $d\iota_v \Pi + \iota_v d\pi = \iota_v \Pi$ is a closed n-form.
Thus, $d\phi_v - \iota_v \Pi \in \mathcal{I}$ and closed in $\bigwedge^n \mathcal{I}$.
But $H^n(\mathcal{I}) = 0$. Indeed, if $\psi = \theta \wedge \alpha + d\theta \wedge \beta$ is closed n-form in $\bigwedge^n \mathcal{I}$, $d\theta \wedge (\alpha + d\beta) = 0(\text{mod} I)$.
The linear algebra mod θ implies $\alpha + d\beta = 0(\text{mod } I)$. This shows

$$\psi = d(\theta \wedge \beta) + \theta \wedge (\alpha + d\beta) = d(\theta \wedge \beta) \quad i.e. [\psi] = 0.$$

For some ε, $d\phi_v - \iota_v \Pi = d\varepsilon$ and $[d(\phi_v] = \iota_v \eta$.

5 Example : Minimal surfaces

Our contact manifold is

$$\begin{aligned} M^{2n+1} &= \{(x,H)|x \in \mathbb{E}^{n+1},\ H \subset T_x\mathbb{E}^{n+1} : \text{oriented hyperplane}\} \\ &= \{(x,e_0)|e_0 \perp H^n, \text{unit tnagent vector}\} \end{aligned}$$

$\pi : M^{2n+1} \to \mathbb{E}^{n+1}$: tangent n-plane bundle ($\cong G_n(T\mathbb{E}^{n+1}) \cong \mathbb{E}^{n+1} \times S^n$).

Given

$$N \hookrightarrow \mathbb{E}^{n+1} : \quad \text{immersed hypersurface}$$

its lift to M is a Legendre submanifold:

$$N \to M^{2n+1} : \quad y \to (y, e_0(y)), \quad \text{(Legendrian lift)}.$$

contact form on M^{2n+1}

At $(x, e_0) \in M$, the *contact form* θ is defined (as usual)

$$\theta_{(x,e_0)}(v) = \langle e_0 | T\pi_{(x,e_0)}(v) \rangle, \qquad v \in T_{(x,e_0)}M.$$

Using orthonormal frames $(e_0, e_1, \ldots, e_n)$ and its dual coframe $(\omega^0, \omega^1, \ldots, \omega^n)$, we have $\theta = \omega^0$ and

$$d\theta = \sum_{i=1}^{n} \pi_i \wedge \omega^i, \quad \pi_i : 1\text{--form},$$
$$\theta \wedge (d\theta)^n \neq 0.$$

As a Langrangian n-form on M, we take

$$\Lambda = \omega^1 \wedge \cdots \wedge \omega^n.$$

This form at $(x, e_0) \in M$ is the volume form Ω of $\mathbb{E}^{n+1}$ contracted by e_0 considered at (x, e_0).

With these data, we have a fuctional on compact Legendrian submanifolds:

$$F_\Lambda(N) = \int_N \Lambda.$$

Since we have $d\omega^i = -\sum_{j=0}^n \omega_j^i \wedge \omega^j$ and $\pi_i = \omega_i^0$

$$d\Lambda = -\pi_1 \wedge \theta \wedge \omega^2 \wedge \cdots \wedge \omega^n - \omega^1 \wedge \pi_2 \wedge \theta \wedge + \cdots$$
$$= \theta \wedge \left(\sum_{i=1}^n \pi_i \wedge \omega_{(i)} \right), \qquad \omega_{(i)} = (-1)^{i-1} \omega^1 \wedge \cdots \wedge \hat{\omega^i} \wedge \cdots \wedge \omega^n$$

This is the P-C form ($\theta \wedge \Pi = 0$) and $\Psi = \sum_{i=1}^n \pi_i \wedge \omega_{(i)}$:

$$\mathcal{E}_\Lambda = \{\theta, d\theta, \Psi = \sum_{i=1}^n \pi_i \wedge \omega_{(i)}\}.$$

On the extremal Legendrian submanifold $d\theta = \sum_{i=1}^n \pi_i \wedge \omega^i = 0$ and by a Cartan's lemma $\pi_i = \sum_{i=1}^n h_{ij} \omega^j, \quad (h_{ij} = h_{ji})$. Also

$$0 = \Psi|_N = \sum_{i=1}^n \pi_i \wedge \omega_{(i)} = \sum_{i,j=1}^n h_{ij} \omega^j \wedge \omega_{(i)} = \left(\sum_{i=1}^n h_{ii} \right) \omega^1 \wedge \cdots \wedge \omega^n.$$

Thus, N is minimal $\Leftrightarrow \sum_{i=1}^n h_{ii} = 0$. (h_{ij} the second fundamental form of N).

conservation law for translation

$\boxed{\phi_v = -\iota_v\Lambda + \theta(v)\beta + \gamma}$.

(a) v: a translation vector field on $\mathbb{E}^{n+1}$ prolonged to M .

(b) $\mathcal{L}_v\Lambda = 0,(\Rightarrow \gamma = 0)$, $d\Lambda = \Pi,(\Rightarrow \beta = 0)$.

Then conservation law for v is $\boxed{\phi_v = \iota_v\Lambda}$, $\qquad (\Lambda = \omega^1 \wedge \cdots \wedge \omega^n)$.
Translation vector v is written on M is written as

$$v = A^0 e_0 + \cdots + A^n e_n$$

where A^a is changes so that v is constant as the frame $(e_0, e_1, \ldots, e_n)$ moves along the fiber.
At $(x, e_0) \in M$, $\iota_v\Lambda$ is

$$A^1\omega^2 \wedge \cdots \wedge \omega^n - A^2\omega^1 \wedge \hat{\omega}^2 \wedge \cdots \wedge \omega^n + \cdots = \sum_{i=1}^{n} A^i\omega_{(i)}.$$

This form expressed as follows:

For $M \ni (x, e_0)$,

$$\begin{aligned} dx =& e_0\omega^0 + \cdots + e_n\omega^n \overset{proj.}{\to} e_1\omega^1 + \cdots + e_n\omega^n \\ &\overset{*}{\to} e_1\omega_{(1)} + \cdots + e_n\omega_{(n)} \qquad (*\text{-operator w.r.t. } \omega) \end{aligned}$$

Then $*dx$ is a vector valued $(n-1)$-form and

$$\langle v, *dx \rangle = \sum_{i=1}^{n} A^i \omega_{(i)} = \iota_v \Lambda.$$

Thus, essentially $*dx$ is the conservation law. $d * dx = 0$ on a minimal hypersurface means coordinate functions are harmonic.
For $n = 2$, especially, $*dx = dy$ (locally) $\exists y : N \to \mathbb{E}^3$ and the map $x + iy : N \to \mathbb{C}^3$ satisfies Cauchy-Riemann eq.

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