On the existence of Sasaki-Einstein metrics

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References

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§1. **Introduction**

1997 J. Maldacena

AdS/CFT correspondence

IIB string theory on $AdS_5 \times Y_5$

(Y_5 : 5-dim. Sasaki-Einstein)

$\Updownarrow$

$N = 1$ 4-dim. superconformal field theory

Def 1.

(M, g) : *Riemannian mfd.*

$(C(M) = \mathbb{R}_+ \times M, \bar{g} = dr^2 + r^2 g)$: *Riemannian cone of* (M, g)

(M, g) *is a* **Sasaki manifold** $\Longleftrightarrow$ $(C(M), \bar{g})$ *is Kähler*

(toric) (*toric*)

Prop 2.

(M^{2m+1}, g) : *Sasaki-Einstein* $\quad$ *($\Rightarrow Ric_g = 2mg$)*

$\Longleftrightarrow$ $(C(M), \bar{g})$: *Ricci-flat Kähler*

Thm 3 (Yau, 1978).

(V, J, g) : compact *Kähler manifold,* $c_1(V) = 0$

$\Rightarrow^{\exists 1}$*Ricci-flat Kähler metric in each Kähler class*

Q.

Noncompact case?

$\bigcup$

$^{\exists}$**Sasaki-Einstein metric?**

Physicists

$$c_1(C(M)) = 0 \Rightarrow^{\exists} \text{S-E metric} \quad \cdots \quad (\star)$$

2006

- Futaki-O.-Wang $\quad C(M)$: toric $\Rightarrow (\star)$ is right

- Gauntlett-Martelli-Sparks-Yau
  In general, $(\star)$ is wrong!! There are obstructions.

§2. Sasaki manifolds

Def 4.

(M, g) : *Sasaki mfd.*, $(C(M), \bar{g}, J)$: *Kähler cone*

$$\tilde{\xi} = Jr\frac{\partial}{\partial r} : \textbf{Reeb vector field}$$

- $\tilde{\xi} - iJ\tilde{\xi}$ *is holomorphic*
- $\tilde{\xi}$ *is Killing.* $\bar{g}(\tilde{\xi}, \tilde{\xi}) = r^2$
- $\xi = \tilde{\xi}_{|\{r=1\}}$ *is a Killing vector field on* $\{r = 1\} \simeq (M, g)$
  $|\xi| = 1$

Exmaple : Prequantization bundle (regular Sasaki manifold)

$(V, J, \underline{g})$: Kähler, $\dim_{\mathbb{C}} = m$ s.t. $[\omega] \in H^2(V, \mathbb{Z})$

$$\Downarrow$$

$p : P \to V$: S^1-bundle, θ : connection s.t. $d\theta = 2\pi\sqrt{-1}p^*\omega$,

$$\Downarrow$$

(P, g) is a Sasaki manifold, where $g = p^*\underline{g} + \eta \otimes \eta$, $\eta = \frac{\theta}{\pi\sqrt{-1}}$,

(Reeb vector field)$= C$(the generator of the S^1-action)

$$\rho = (2m+2)\omega \iff g : \text{ regular S-E}$$

Def 5.

(M, g) : *Sasaki mfd.*

$|\xi| = 1 \Rightarrow \ \mathcal{F}_\xi$: 1-*dim.foliation on* M **Reeb foliation**

• *Sasaki structure* $(M, g; \xi, \eta, \Phi)$ *induces a transverse Kähler structure of* $\mathcal{F}_\xi$

• (M, g) *is Sasaki-Einstein if and only if the transverse Kähler structure satisfies* $\rho^T = (2m + 2)\omega^T$

§3. Spectrum of Reeb vector field

Def 6.

(M,g) : *Sasaki mfd.*

$$H(M)$$
$$:= L^2\text{-closure of } \left\{ f \in C^\infty(M) \,\middle|\, \begin{array}{c} {}^\exists \tilde{f} : \ \textit{hol.on } \{r \leq 1\} \textit{ s.t.} \\ \tilde{f}_{|\{r=1\}} = f, \tilde{f} \to 0 \ (r \to 0) \end{array} \right\}$$

: **Hardy space**

Def 7. $T = \frac{\xi_{|H(M)}}{\sqrt{-1}} : H(M) \to H(M)$

• T is a first order self-adjoint Toeplitz operator which has a positive symbol.

$$\Downarrow$$

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow \infty$$

: discrete spectrum (**charge** of (S, ξ, Φ))

• Charges depends only on ξ and the transverse holomorphic structure.

Example

$(V, L, \underline{g})$: polarized Kähler manifold

$\Rightarrow M = S(L^*)$: regular Sasaki

$$H(M) \simeq \bigoplus_{k=0}^{\infty} H^0(V; L^k),$$

$H^0(V; L^k)$: charge k-eigenspace

Thm 8. (M-S-Y, Boutet de Monvel-Guillemin)

(M^{2m+1}, g) : *cpt. Sasaki mfd.*

$$\mathrm{Vol}(M,g) = \gamma_{2m+1} \lim_{t \searrow 0} t^{m+1} \sum_{j=1}^{\infty} e^{-t\lambda_j}$$

$$\gamma_{2m+1} = \mathrm{Vol}(S^{2m+1}(1))$$

- $\mathrm{Vol}(M,g)$ *is an invariant of* (M, ξ, Φ)

Thm 9.

$(M^{2m+1}, g; \xi, \eta, \Phi)$: *cpt. Sasaki mfd.,* λ : *charge of* (M, ξ, Φ)

$$\Rightarrow \lambda(\lambda + 2m) : \textit{an eigenvalue of } \Delta_{(M,g)}$$

§4. **Obstructions**

Question

$\exists$Ricci-flat Kähler cone metric on $C(M)$?

$$\Updownarrow$$

$\exists$S-E metric on M?

$$\Updownarrow$$

$\exists$transverse K-E metric s.t. $\rho^T = (2m+2)\omega^T$?

charge $\Rightarrow$ obstructions

① **Volume minimization (M-S-Y)**

If $^{\exists}g_0$ S-E metric. Then ξ_{g_0} is a critical point of "volume function" :

$$\text{Vol} : \mathbf{Reeb}_c \to \mathbb{R}$$

Thm 10 (M-S-Y, F-O-W).

$$d_\xi \text{Vol} = CF_\xi, \quad F_\xi : \textit{Sasaki-Futaki inv.}$$

Thm 11 (F-O-W, C-F-O).

$C(M)$: *toric,* $c_1(C(M)) = 0$

- $^{\exists 1}\xi_{min} \in \mathbf{Reeb}_c$: *minimizer of* Vol
- $^{\exists}g_0$: *S-E metric on* M *s.t.* $\xi_{g_0} = \xi_{min}$
- *For any* $\xi_{min} \neq \xi \in \mathbf{Reeb}_c$, $^{\nexists}$*S-E metric whose Reeb vector field is* ξ.
- g_0 *is unique up to "transverse automorphisms".*

② Bishop obstruction (G-M-S-Y)

Thm 12 (Bishop, 1964).

(M^m, g) : *cpt. Riem. mfd.*, $Ric \geq (m-1)g$

$$\Rightarrow \text{Vol}(M, g) \leq \gamma_m \ (\text{“}=\text{”} \Longleftrightarrow (M, g) \simeq S^m(1))$$

Bishop obstruction

$(M^{2m+1}, g_0; \xi, \eta, \Phi)$ is a compact S-E mfd.

$$\Rightarrow \text{Vol}(\xi, \Phi) \leq \gamma_{2m+1}.$$

③ Lichnerowicz obstruction (G-M-S-Y)

Thm 13 (Lichnerowicz,Obata,1958).

(M^m, g) : *complete Riemannian mfd.*, $Ric \geq (m-1)g$

$\Rightarrow M$: *compact*, $\lambda_1(M,g) \geq m$ ("=" $\iff$ $(M,g) \simeq S^m(1)$)

Lichnerowicz obstruction

$(M^{2m+1}, g_0; \xi, \eta, \Phi)$ is a compact S-E mfd.

$\Rightarrow$The first charge λ_1 of (M, ξ, Φ) satisfies $\lambda_1 \geq 1$.

Example : links of isolated hypersurface singularities

$$\mathbb{C}^* \curvearrowright \mathbb{C}^{m+2},\ (z_0, \cdots, z_{m+1}) \to (q^{w_0} z_0, \cdots, q^{w_{m+1}} z_{m+1})$$

$$q \in \mathbb{C}^*, \quad (w_0, \cdots, w_{m+1}) \in \mathbb{N}^{m+2}$$

F : polynomial s.t.

- $F(q \cdot z) = q^d F(z) \quad d \in \mathbb{N}$
- $\mathrm{Sing}X = \{0\} \in \mathbb{C}^{m+2},\ (\mathbb{C}^* \curvearrowright X = \{z \in \mathbb{C}^{m+2} | F(z) = 0\})$
- $|w| := \sum w_j > d \quad (\iff$ "log Faноness" of $X/\mathbb{C}^*)$

Lem 14.

ζ : *generator of* $S^1 \subset \mathbb{C}^*$ *action*

$$\Rightarrow \xi := \frac{m+1}{|w|-d}\zeta \in \mathbf{Reeb}_c$$

Prop 15.

$$\lambda_1(\xi) = \frac{(m+1)\min\{w_j\}}{|w|-d}$$

$$\mathrm{Vol}(\xi) = \frac{d\gamma_{2m+1}(|w|-d)^{m+1}}{(m+1)^{m+1}\prod w_j}$$

$$F(z) := z_0^{a_0} + \cdots + z_{m+1}^{a_{m+1}}, \ {}^{\forall} a_j w_j = d$$

- $|w| - d > 0 \iff \sum \left(\frac{1}{a_j}\right) > 1$
- $\mathrm{Vol}(\xi) \le \gamma_{2m+1} \iff \left(\prod a_j\right) \left(\sum \left(\frac{1}{a_j}\right) - 1\right) \le (m+1)^{m+1}$
- $\lambda_1(\xi) \ge 1 \iff (m+1) \min \left\{\frac{1}{a_j}\right\} \ge \sum \left(\frac{1}{a_j}\right) - 1$

§5. Problems

Kähler case

Conj. (Donaldson, Tian)

Let (M, L) be a polorized manifold. Then

$$^{\exists}\text{constant scalar curvature Kähler metric in } c_1(L)$$
$$\Updownarrow$$
$$(M, L) \text{ is K-polystable}$$

- asymptotic behavior of Bergman kernel

- balanced metric etc.

$$\Downarrow$$

Sasaki case??