The non-acyclic Reidemeister torsion for knots

Yoshikazu Yamaguchi

Geometry for Quantization 2007
at Waseda University
2007.9.6

1. Notations & Definitions

2. The non-acyclic R-torsion and the twisted Alexander invariant

3. Critical points of the non-acyclic R-torsion

1 Notations&Definitions

Twisted chain complex

$K \subset S^3$: a knot,

$E_K = S^3 \setminus N(K)$: the knot exterior,

$\rho : \pi_1(E_K) \to \mathrm{SU}(2)$ a homomorphism.

Define

$$C_*^\rho(E_K)$$

$$= C_*(\widetilde{E_K};\mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(E_K)]} \mathfrak{su}(2)_\rho$$

Here

$\widetilde{E_K}$: the universal cover of E_K,

$C_*(\widetilde{E_K};\mathbb{Z})$: the cell complex of $\widetilde{E_K}$, which consists of $\mathbb{Z}[\pi_1(E_K)]$-modules,

$\mathfrak{su}(2)_\rho$: $\mathbb{Z}[\pi_1(E_K)]$-module via $Ad \circ \rho$

$$Ad\colon \mathrm{SU}(2) \longrightarrow \mathrm{Aut}(\mathfrak{su}(2)),$$
$$A \mapsto (v \mapsto AvA^{-1}),$$

This is called "the $\mathfrak{su}(2)_\rho$-twisted chain complex".

$H_*^\rho(E_K)$ denotes the homology group.

- $X(E_K)$: the $\mathrm{SU}(2)$-character variety of $\pi_1(E_K)$.

- $X(E_K) \simeq$
  $Hom(\pi_1(E_K), \mathrm{SU}(2))/conj$

- $H_*^\rho(E_K) \simeq H_*(\pi_1(E_K); \mathfrak{su}(2)_\rho)$
  Moreover
  $H_1^\rho(E_K) \simeq$
  $T_{[\rho]}^{zar}(Hom(\pi_1(E_K), \mathrm{SU}(2))/conj)^*$

Regularity for representations

Roughly, with a notion of regularity there is a canonical way to choose bases for twisted homologies.

$\rho : \pi_1(E_K) \longrightarrow \mathrm{SU}(2)$ is regular
if ρ is irreducible and $\dim H_1^\rho(E_K) = 1$.

For a regular representation ρ, we have

$$\dim H_1^\rho(E_K) = \dim H_2^\rho(E_K) = 1$$

and

$$H_j^\rho(E_K) = 0 \text{ for all } j \neq 1, 2.$$

λ : the preferred longitude of K.

(i.e., $lk(K, \lambda) = 0$.)

A regular representation ρ is λ-regular (J. Porti 1997) if the inclusion $\iota\colon \lambda \hookrightarrow E_K$ induces a surjective map

$$\begin{aligned} \iota_*\colon H_1^{\rho}(\lambda) &\to H_1^{\rho}(E_K) \\ [\lambda \otimes P^{\rho}] &\mapsto \iota_*([\lambda \otimes P^{\rho}]) \neq 0 \end{aligned}$$

where P^{ρ} in $\mathfrak{su}(2)$ such that $Ad_{\rho(\lambda)}(P^{\rho}) = P^{\rho}$.

The following fact is known :

For a λ-regular representation ρ,

$$H_1^\rho(E_K) = \mathbb{R}[\lambda \otimes P^\rho] \text{ and}$$
$$H_2^\rho(E_K) = \mathbb{R}[\partial E_K \otimes P^\rho]$$

where P^ρ is a vector in $\mathfrak{su}(2)$ such that $Ad_{\rho(\gamma)}(P^\rho) = P^\rho$ for $\forall \gamma \in \pi_1(\partial E_K)$.

The non-acyclic R-torsion for K

We assume that

a representation ρ is λ-regular.

Let $d_i : C_i^{\rho}(E_K) \longrightarrow C_{i-1}^{\rho}(E_K)$ and

- $B_{i-1} = \mathrm{Im}(d_i)$,
- $Z_i = \ker(d_i)$,

- $\mathrm{b}^i \subset C_i^\rho(E_K)$: vectors such that $d_i(\mathrm{b}^i)$ is a basis of B_{i-1},
- h^i : the following basis of H_i^ρ
  $\mathrm{h}^1 = [\lambda \otimes P^\rho], \quad \mathrm{h}^2 = [\partial E_K \otimes P^\rho]$
  and $\mathrm{h}^i = \emptyset$ for $i \neq 1, 2$ and
- $\widetilde{\mathrm{h}}^i$: a lift of h^i in Z_i.

Then

$$\begin{aligned} C_i^\rho(E_K) &= Z_i \oplus \widetilde{B}_i \\ &= d_{i+1}(\widetilde{B}_{i+1}) \oplus \widetilde{H_i^\rho} \oplus \widetilde{B_i} \end{aligned}$$

where

$\widetilde{B}_i$ is a lift of B_{i-1}, i.e., $d_i(\widetilde{B}_i) = B_{i-1}$.

Then the non-acyclic R-torsion $\mathbb{T}^{\boldsymbol{K}}_{\boldsymbol{\lambda}}(\boldsymbol{\rho})$ for $\boldsymbol{K}$ and $\boldsymbol{\rho}$ is defined by the following alternative product of the determinants of base change matrices

$$\mathbb{T}^{\boldsymbol{K}}_{\boldsymbol{\lambda}}(\boldsymbol{\rho}) = \boldsymbol{\varepsilon} \cdot \prod_{i=0}^{n} [\boldsymbol{d}_{i+1}(\mathbf{b}^{i+1})\widetilde{\mathbf{h}}^{i}\mathbf{b}^{i}/\mathbf{c}^{i}_{\boldsymbol{\mathcal{B}}}]^{(-1)^{i+1}}$$

where

- $[\mathrm{b}/\mathrm{a}]$: the $\det$ of the base change matrix from a to b,
- $\mathbf{c}^i_{\mathcal{B}}$: a basis of $C^\rho_i(E_K)$ given by i-dimensional cells of E_K and a basis $\mathcal{B}$ of $\mathfrak{su}(2)$,
- ε : a sign defined by an orientation of $H_*(E_K;\mathbb{R})$.

2 $\mathbb{T}_\lambda^K$ and the twisted Alexander invariant

There exists a relation between $\mathbb{T}_\lambda^K(\rho)$ and the twisted Alexander invariant $\Delta_{K,Ad\circ\rho}$.

Proposition 1. If ρ is $\boldsymbol{\lambda}$-regular, then the twisted Alexander invariant $\boldsymbol{\Delta}_{K,Ad\circ\rho}(t)$ can be defined.

Theorem 2. If ρ is $\boldsymbol{\lambda}$-regular, then

$$\mathbb{T}_{\boldsymbol{\lambda}}^{K}(\rho) = -\varepsilon \, \frac{d}{dt} \boldsymbol{\Delta}_{K,Ad\circ\rho}(t) \Big|_{t=1} .$$

Review of $\Delta_{K,Ad\circ\rho}(t)$

$$\pi_1(E_K) = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_{k-1} \rangle$$

: Wirtinger presentation

$$\alpha : \pi_1(E_K) \to H_1(E_K; \mathbb{Z}) \simeq \mathbb{Z} = \langle t \rangle$$
$$\mu \mapsto t$$

is an abelianization homomorphism where μ is a meridian and $\langle t \rangle$ is a multiplicative group .

We choose a basis of $\mathfrak{su}(2)$, for example

$$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Let

$$\Phi : \mathbb{Z}[\pi_1(E_K)] \longrightarrow M_3(\mathbb{C}[t, t^{-1}])$$

$$\sum_i a_i \cdot \gamma_i \mapsto \sum_i a_i \cdot Ad_{\rho(\gamma_i)} \alpha(\gamma_i)$$

where a_i is an integer and $Ad_{\rho(\gamma_i)}$ is a matrix w.r.t the basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

$A^j_{K,Ad\circ\rho}$: the matrix by deleting the j-th 3 rows from $3k \times 3(k-1)$-matrix $\Phi\left(\frac{\partial r_i}{\partial x_j}\right)_{1\le i\le k-1, 1\le j\le k}$.

(these differentials are Fox differentials)

If $\exists j$ s.t. $\det(A^j_{K,Ad\circ\rho}) \neq 0$, then the twisted Alexander invariant $\Delta_{K,Ad\circ\rho}(t)$ is defined by

$$\Delta_{K,Ad\circ\rho}(t) = \frac{\det(A^j_{K,Ad\circ\rho})}{\det(\Phi(x_j - 1))}.$$

Example

K:figure eight knot

$$\pi_1(E_K) = \langle x, y \mid wx = yw \rangle$$

where $w = [x^{-1}, y]$.
By Riley's method,

$$\rho : \pi_1(E_K) \longrightarrow \mathbf{SU(2)}$$

$$x \longmapsto \begin{pmatrix} \sqrt{s} & \frac{1}{\sqrt{s}} \\ 0 & \frac{1}{\sqrt{s}} \end{pmatrix}$$

$$y \longmapsto \begin{pmatrix} \sqrt{s} & 0 \\ -u\sqrt{s} & \frac{1}{\sqrt{s}} \end{pmatrix}$$

(s, u) satisfy that

$$u^2 + \left(3 - \left(s + \frac{1}{s}\right)\right)(u+1) = 0$$

Then

$$\Delta_{K,Ad\circ\rho}(t) = \frac{\det \Phi(\frac{\partial}{\partial x} wxw^{-1}y^{-1})}{\det \Phi(y-1)}$$

Moreover the numerator of $\frac{d}{dt}\boldsymbol{\Delta}_{K,Ad\circ\rho}(t)\Big|_{t=1}$ is given by

$$\frac{1}{s^2}\left(-1+2\,u+u^2+\right.$$
$$s^4\,\left(-1+2\,u+u^2\right)-$$
$$s\,\left(-3+6\,u+6\,u^2+2\,u^3\right)-$$
$$s^3\,\left(-3+6\,u+6\,u^2+2\,u^3\right)+$$
$$\left.s^2\,\left(-7+4\,u+7\,u^2+4\,u^3+u^4\right)\right),$$

the denominator of $\frac{d}{dt}\Delta_{K,Ad\circ\rho}(t)\Big|_{t=1}$ is given by $-(s+\frac{1}{s}-2)$.

Finally, we have

$$\mathbb{T}^{K}_{\lambda}(\rho) = -\varepsilon\cdot\left(2\left(s+\frac{1}{s}\right)-1\right)$$

by using the equation $u^2-\left(3-\left(s+\frac{1}{s}\right)\right)(u+1)=0$.

Since $\mathrm{tr}\,\rho(\mu) = \sqrt{s} + \frac{1}{\sqrt{s}}$,
we can also express

$$\mathbb{T}^{K}_{\lambda}(\rho) = -\varepsilon \cdot \left(2\mathrm{tr}^{\,2}\rho(\mu) - 5\right).$$

Fact

K : figure eight knot

$\mathrm{tr}\,\rho(\mu)$ gives a local parameter on the $\mathrm{SU}(2)$-character variety.

Therefore

$\mathbb{T}^K_\lambda(\rho) = -\varepsilon \cdot \big(2(\mathrm{tr}\,\rho(\mu))^2 - 5\big)$ has critical points at $\mathrm{tr}\,\rho(\mu) = 0$.

3 Critical points of $\mathbb{T}_{\lambda}^{K}$

Binary dihedral representations

An $\mathrm{SU}(2)$-representation ρ is a binary dihedral representation if

$$\rho(x_j) = \left(\cos\theta_j + \sin\theta_j \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$$

for generators of Wirtinger presentation
$\pi_1(E_K) = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_{k-1} \rangle$

The non-acyclic R-torsion $\mathbb{T}^K_\lambda(\rho)$ does not change by taking conjugation of ρ.

We can regard $\mathbb{T}^K_\lambda$ as a function on the $\mathrm{SU}(2)$-character variety of a knot group $\pi_1(E_K)$.

Hereafter, K is a two–bridge knot in S^3.

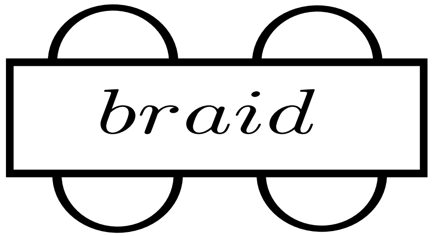

Figure 1: Two bridge knot

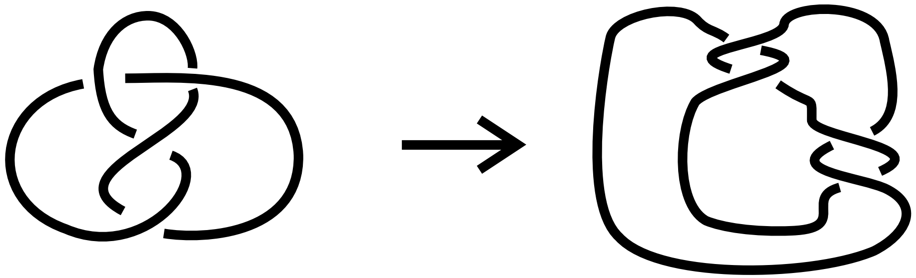

Figure 2: Deformation of figure eight knot

Proposition 3. If the λ-regular component of the $\mathrm{SU}(2)$-character variety of K contains the characters of binary dihedral representations, then $\mathbb{T}_{\lambda}^{K}$ has a critical point at the character of each binary dihedral representation.

Lemma 4. K is a two–bridge knot. Every non-abelian $\mathbf{SU(2)}$-representation ρ such that $\operatorname{tr}\rho(\mu) = 0$ is conjugate to a binary dihedral representation.

For the character χ_ρ of each binary dihedral representation ρ such that

$$\chi_\rho \in \exists U_\rho \simeq (0, 1) \quad \text{diffeo}$$

and $I_\mu : U_\rho \ni \chi_\tau \mapsto \operatorname{tr}\tau(\mu) \in \mathbb{R}$ gives a local parameter.

This Lemma follows from the results of

G. Burde,

"$\mathbf{SU(2)}$-representation spaces for two-bridged knot groups"

and

M. Heusener and E. Klassen,

"Deformations of dihedral representations ".

Proof of Proposition. On U_ρ, a local parameter $I_\mu(\chi_\tau)$ is expressed as $2\cos\theta$. By the relation between $\mathbb{T}_\lambda^K(\tau)$ and $\Delta_{K,Ad\circ\tau}(t)$,

$$\mathbb{T}_\lambda^K(\tau) = \frac{f(\cos 2\theta)}{2\cos 2\theta - 2}$$

where f is a smooth function of $\cos 2\theta$. Moreover

$$\mathrm{tr}\,\rho(\mu) = 0 \Rightarrow \frac{d}{d\theta}\mathbb{T}_\lambda^K = 0.$$

□