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Index of the Dirac operator on S^4 and the infinite dimensional Grassmannian on S^3

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0. Introduction

In this paper we study the null space of the massless Dirac opertor on a hemisphere of S^4 with boundary condition given on the equator $\simeq S^3$. The boundary conditions we consider are nonlocal ones that are realized by the infinite dimensional Grassmannian associated to the Dirac operator on the equator, called Hamiltonian. We shall introduce in section 6 the infinite dimensional Grassmannian $Gr(H)$ of the Hilbert space H of square integrable even spinors on S^3. H is polarized by the spaces of eigenvectors $H_{\pm}$ for positive (resp. negative) eigenvalues of the chirality preserving Dirac operator (Hamiltonian) on S^3. $Gr(H)$ is defined by this polarization. Let $D_{W^{\perp}}$ be the Dirac operator acting on even spinors on a hemisphere whose boundary values on the equator are in a given $W \in Gr(H)$. We shall show in Theorem 6.4 that

$$index D_{W^{\perp}} = virtual \dim W. \tag{0-1}$$

In particular, for $W = H_-$ both sides are 0; $W = H_-$ corresponds to the so-called Atiyah-Patodi-Singer boundary condition [A-P-S, B-W 1]. This type of index formula for a general manifold with boundary was discussed in [Wo, B-W 1, B-W 2]. In [Wi] Witten gave a brief discussion on the infinite dimensional Grassmannian on S^1 as nonlocal boundary conditions of the $\overline{\partial}$-operator on the disc. Recall that on $C^1 \subset S^2$ the Dirac operator is reduced to the $\overline{\partial}$-operator and the Hamiltonian on the equator S^1 is $\frac{\partial}{\partial\theta}$. The Hilbert space H on the equator is polarized by $H_+ = \{e^{in\theta};\, n \geq 1\}$ and $H_- = \{e^{-in\theta};\, n \geq 0\}$. Now H_+ (resp. H_-) is the set of boundary values of holomorphic functions on the disc (resp. outside of the disc). In the same fashion we shall investigate the problem of extending the spinors on the equator S^3 to zero mode spinors on each hemisphere. Let $H_{\pm}$ be as mentioned earlier the space of the eigenvectors of Hamiltonian on $B = \{z \in C^2;\, |z| = 1\} \simeq S^3$ that correspond to the positive (resp. negative) eigenvalues. Then in Theorem 5.10 we show that

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H_+ is the space of the boundary traces of zero mode spinors of even chirality on the north hemisphere and H_- is the space of those spinors on the south hemisphere. The corresponding assertion for the boundary traces of zero mode spinors of odd chirality on both hemispheres are also discussed. By virtue of Theorem 5.10 the proof of index formula (0-1) reduces to a problem on linear algebra. This extension property is not valid in general, [B-W 1, §14], nor the validity of (0-1) on a general manifold with boundary. Now we shall give an overview of the sections.

In section 2 we introduce basic ingredients on our *membrane* M. A membrane M is a smooth manifold homeomorphic to S^4 obtained by patching $\boldsymbol{C}^2_z$ and $\widehat{\boldsymbol{C}}^2_w$ together by the transition function $w = v(z) = -\frac{\bar{z}}{|z|^2}$. In section 3 formulas familiar in Riemannian geometry are described in their concrete and explicitly expressed forms. The Levi-Civita connection on M is given by the gauge potentials $\frac{|z|^2}{1+|z|^2}\sigma(z)^{-1}(d\sigma)_z$, here $\sigma = |z|^2 v_*$, on each hemisphere. The explicit coordinate representation of the connection enables us to obtain the spinor representation of Levi-Civita connection lifted to the Spin bundle of M. A new aspect on the Levi-Civita connection on $B \simeq S^3$ that relates to the C-R (Cauchy-Riemann) structure on B is proposed.

In section 4 we shall first give a coordinate representation of the Dirac operator on M, which is useful for the later calculations. On the equator B there exists a chirality preserving Dirac operator (Hamiltonian) $\mathcal{P}$. Let D be the Dirac operator on M restricted to act on the space of even spinors S^+ ; $D = \mathcal{D}|S^+$, and let $\mathcal{P}$ act on $S^+|B$. Then we have the decomposition of D to the radial part and the longitudinal part; $D = \gamma_0(\mathbf{n} - \mathcal{P})$. Correspondingly on the spinors of odd chirality we have $D^\dagger = (\mathbf{n} + \mathcal{P})\gamma_0$. $\mathcal{P}$ has a simple expression in terms of the C.R. basis on B, (4-3-2).

In section 5 we shall calculate the eigenvalues of $\mathcal{P}$ and obtain a complete orthogonal system of eigenvectors in explicit forms.

THEOREM 5.3. *The eigenvalues of* $\mathcal{P}$ *are*

$$\pm\left(\frac{3}{2} + r\right) ; \quad r = 0, 1, 2, \cdots$$

with multiplicity $(r+1)(r+2)$, *in particular, there is no zero mode spinor of* $\mathcal{P}$ *and the spectrum is symmetric relative to* 0.

The same problem but for another metric on S^3 was studied in [Hi, Sch] and the author was guided by [Hi, Gol, Sch], though the author's method provides a new way of finding systematically the eigenvectors of Hamiltonian, (section 5.1). In 5.2 we discuss the extension problem of spinors that we mentioned above:

THEOREM 5.10. *The boundary trace* b *gives the following isomorphisms*

$$\{\varphi \in H^1(\{|z| \leq 1\}, S^+) : D\varphi = 0 \text{ on } |z| < 1\} \xrightarrow[b]{\simeq} H_+ \cap H^{\frac{1}{2}}(B, S^+)$$

$$\{\hat{\varphi} \in H^1(\{|w| \leq 1\}, S^+) : D\hat{\varphi} = 0 \text{ on } |w| < 1\} \xrightarrow[b]{\simeq} H_- \cap H^{\frac{1}{2}}(B, S^+) .$$

The procedure is analogous to the construction of solutions to Laplace equation in $\boldsymbol{R}^3$ by the method of separation of variables from eigenvectors of Laplace-Beltrami operator on S^2 and a radial differential operator.

Subsection 6.1 is devoted to the discussion on the infinite dimensional Grassmannian $Gr(H)$. We shall discuss in the rest the index of Dirac operator D on the hemisphere with boundary condition. For a $W \in Gr(H)$, the operator D has the closed extension $D_{W^\perp}$ with the domain of definition consisting of those spinors on the unit ball whose boundary values lie in W. To have the realization $D_{W^\perp}$ and to show that its adjoint is equal to $D_W^\dagger$ we shall give in Proposition 6.2 the explicit description of a bounded inverse for $D_{W^\perp}$. Here we use the canonical basis of W. Proposition 6.2 and Theorem 5.10 are the keys in our discussion. Then we prove the index theorem (0-1). The general A-P-S boundary value problem can be formulated equally well on a general manifold with boundary. But we lack necessary analysis for the general case where the metric is not cylindrical near the boundary; for example, we don't know if the adjoint operator of $D_{W^\perp}$ is $D_W^\dagger$, nor can we expect the counterpart of our extension properties. Recently Grubb [Gr] and Gilkey [Gil-2] extended the index formula for the boundary condition of Atiyah-Patodi-Singer type ($W = H_-$) to the case where the structures near the boundary are not product. We note also that the general results of [Se, B-W 1] use the Grassmannian of pseudo-differential projections with the same symbol. In their cases the result corresponding to Proposition 6.2 is assured by the theory of pseudo differential operator. Our proofs are performed by direct calculations on the membrane.

The index theorem of Dirac operator coupled to a gauge potential was investigated in [A-S, S-2]. In 6.3 we shall discuss briefly that the index theorem of coupled Dirac operator is in some sense subordinate to our index theorem with Grassmannian boundary condition. In fact a $su(N)$–gauge potential defines a Grassmannian element $W_A \in Gr(H^N)$, and then we can show that the index of Dirac operator coupled by A is equal to the *virtual* $\dim W_A$.

I thank Professor B. Booss of Roskilde University for his continuous encouragement. He sent me the draft [B-W 1] before its publication that helped me understand where difficult points are for the general case. Also Professor Booss brought the paper [Sch] to my notice after the first draft of this paper was written in 1990, and by virtue of this I was able to find a mistake in the calculations in 5.1. I thank also Professor K. Furutani of Tokyo Sciences University for his criticism on section 6 of earlier versions; Proposition 6.2 and Proposition 6.3 are added to reply to his objections.

1. Clifford algebra on a complex vector space and subalgebras

As for the facts about Clifford algebra the reader can refer to several good expositions [A-B-Sh, Gil, L-M]. Here we exhibit some formulas that will be cited in this paper emphasizing the manner of reduction to subspaces.

1.1. Let V be a $2n$-dimensional real vector space endowed with the canonical symmetric bilinear form $\langle\, ,\, \rangle$. Let J be a complex structure on V; $J^2 = -1$. J is extended to the complexification V^c. There is a real basis $\{\mathbf{e}_i\}_{i=1}^{2n}$ of V such that $J\mathbf{e}_i = \mathbf{e}_{n+i}$. This gives also the complex basis of V^c. We prefer the basis of V^c given by $\mathbf{f}_i = \frac{1}{2}(\mathbf{e}_i - \sqrt{-1}\mathbf{e}_{n+i})$, $\mathbf{f}_{\bar{i}} = \frac{1}{2}(\mathbf{e}_i + \sqrt{-1}\mathbf{e}_{n+i})$. The real vectors have the form $\sum_i a_i\mathbf{f}_i + \bar{a}_i\mathbf{f}_{\bar{i}}$ with $a_i \in \boldsymbol{C}$. The Clifford algebra $C(V^c)$ of the complex vector space $(V^c, <\ \ >)$ is the algebra over $\boldsymbol{C}$ containing the identity 1 and generated by the vectors $\mathbf{f}_i$, $\mathbf{f}_{\bar{i}}$ with the relations :

$$\mathbf{f}_i\,\mathbf{f}_j + \mathbf{f}_j\,\mathbf{f}_i = \mathbf{f}_{\bar{i}}\,\mathbf{f}_{\bar{j}} + \mathbf{f}_{\bar{j}}\,\mathbf{f}_{\bar{i}} = 0\,, \quad \mathbf{f}_i\,\mathbf{f}_{\bar{j}} + \mathbf{f}_{\bar{j}}\,\mathbf{f}_i = \delta_{ij}.$$

Let G^c be the Lie subgroup of $GL(2n, \boldsymbol{C})$ of those linear transformations on V^c that preserve the bilinear form invariant. G^c is isomorphic to $O(2n, \boldsymbol{C})$. The subspace $\mathcal{G}^c$ spanned by $\mathbf{f}_i\mathbf{f}_{\bar{i}}$, $\mathbf{f}_i\mathbf{f}_j$, $\mathbf{f}_i\mathbf{f}_{\bar{j}}$, $\mathbf{f}_{\bar{i}}\mathbf{f}_j$, $\mathbf{f}_{\bar{i}}\mathbf{f}_{\bar{j}}$, $i < j$, is a Lie subalgebra of $C(V^c)$. If we associate to $X \in \mathcal{G}^c$ the linear map

$$\text{(1-1-1)} \qquad \lambda(X) : v \longrightarrow Xv - vX \qquad v \in V^c,$$

then λ defines a representation of $\mathcal{G}^c$ on V^c. We shall identify $\mathcal{G}^c$ and $\lambda\mathcal{G}^c$. $\mathcal{G}^c$ is isomorphic to $o(2n, \boldsymbol{C})$. There are some canonical morphisms on $C(V^c)$.

(1) The canonical automorphism of $C(V^c)$ is defined as the extension of the linear map $\alpha : V^c \mapsto C(V^c)$, given by $\alpha x = -x$.

(2) The bar operation is the unique antiautomorphism characterlized by $\overline{(x \otimes c)} = \alpha(x) \otimes \bar{c}$ for $x \in V$, $c \in C$.

The real Clifford algebra $C(V)$ of $(V, \langle\, ,\, \rangle)$ is the subalgebra of those elements of $C(V^c)$ that change the sign under the bar operation:

$$\text{(1-1-2)} \qquad C(V) = \left\{ g \in C(V^c); \quad \begin{array}{l} \bar{g}_a + (-1)^{\deg g_a - 1}\, {}^t g_a = 0 \\ \text{for each homogeneous component } g_a \text{ of } g \end{array} \right\}.$$

The real objects are described in this manner.

Let G be the subgroup of G^c of those matrices that preserve V. G is isomorphic to $O(2n)$. The Lie algebra $\mathcal{G}$ of G is given by $\mathcal{G} = \{X \in \mathcal{G}^c;\ XV \subset V\}$. $\mathcal{G} = \mathcal{G}^c \cap C(V)$ and $\mathcal{G}$ is isomorphic to $o(2n)$. We shall fix the notations G and $\mathcal{G}$ throughout this paper.

1.2. For a vector $\mathbf{z} \in V$ let $V_{\mathbf{z}}$ be the subspace of V that is perpendicular to $\mathbf{z}$: $V_{\mathbf{z}} = \{v \in V;\ < v, \mathbf{z} >= 0\}$. We put $G_{\mathbf{z}} = \{A \in G;\ AV_{\mathbf{z}} \subset V_{\mathbf{z}}\}$. $G_{\mathbf{z}}$ is a Lie subgroup of G isomorphic to $O(2n-1)$. The Lie algebra of $G_{\mathbf{z}}$ is $\mathcal{G}_{\mathbf{z}} = \{X \in \mathcal{G}; X\mathbf{z} = 0\}$. $\mathcal{G}_{\mathbf{z}}$ is isomorphic to $o(2n-1)$.

The Clifford algebra $C(V_{\mathbf{z}})$ of $(V_{\mathbf{z}}, < \ >)$ considered as a subalgebra of $C(V)$ is also written as

$$(1\text{-}2\text{-}1) \qquad C(V_{\mathbf{z}}) = \left\{ g \in C(V);\ \begin{matrix} g_a\mathbf{z} + (-1)^{\deg g_a - 1}\mathbf{z}g_a = 0 \\ \text{for each homogeneous component } g_a \text{ of } g \end{matrix} \right\}.$$

We have

$$(1\text{-}2\text{-}2) \qquad G_{\mathbf{z}} = G \cap C(V_{\mathbf{z}}) \qquad \mathcal{G}_{\mathbf{z}} = \mathcal{G} \cap C(V_{\mathbf{z}}).$$

1.3. The universal covering group $Pin(2n)$ of G can be realized as a subgroup of invertible elements of $C(V)$. Corresponding to the Z_2-graduation of $C(V)$, $Pin(2n)$ is decomposed to $Pin^0(2n) \oplus Pin^1(2n)$. $Pin^0(2n) = Spin(2n)$ is the universal covering group of $SG = \{g \in G : \det g = 1\}$. Similarly $Spin(2n-1)$ is the universal covering group of $SG_{\mathbf{z}} = \{g \in G_{\mathbf{z}} : \det g = 1\}$ and we have the relation:

$$(1\text{-}3\text{-}1) \qquad Spin(2n-1) = Spin(2n) \cap C(V_{\mathbf{z}}) = \{g \in Spin(2n) :\ g\mathbf{z} = \mathbf{z}g\}.$$

Obviously $\frac{1}{|z|}\mathbf{z} \in Pin^1(2n)$. This fact is often used in this paper. Clifford algebra $C(V^c)$ decomposes as a direct sum of left $Spin(2n)$ modules in the form: $C(V^c) = 2^n\Delta$. Δ decomposes as the direct sum of two irreducible representations $\Delta^{\pm}$ of dimension 2^{n-1}. We have homomorphisms: $V \otimes \Delta^{\pm} \longrightarrow \Delta^{\mp}$, which come from Clifford multiplication on the left. $Spin(2n-1)$ is a subgroup of $Spin(2n)$ and acting on Δ commutes with the multiplication by $|z|^{-1}\mathbf{z}$. Hence $|z|^{-1}\mathbf{z} : \Delta^+ \longrightarrow \Delta^-$ defines an isomorphism of representation spaces of $Spin(2n-1)$. Δ^+ (resp.Δ^-) is thus an irreducible representation space of $Spin(2n-1)$.

EXAMPLE. We shall write explicit formulas for the case $n = 2$ because we use it in the future. The Chevalley basis of $\mathcal{G}$ are

$$(1\text{-}3\text{-}2) \qquad \begin{aligned} H &= \sqrt{-1}(\mathbf{f}_1\mathbf{f}_{\bar{1}} - \mathbf{f}_{\bar{2}}\mathbf{f}_2), & H' &= \sqrt{-1}(\mathbf{f}_1\mathbf{f}_{\bar{1}} - \mathbf{f}_2\mathbf{f}_{\bar{2}}) \\ A &= \sqrt{-1}(\mathbf{f}_1\mathbf{f}_2 - \mathbf{f}_{\bar{1}}\mathbf{f}_{\bar{2}}), & B &= \sqrt{-1}(\mathbf{f}_1\mathbf{f}_2 + \mathbf{f}_{\bar{1}}\mathbf{f}_{\bar{2}}) \\ C &= \sqrt{-1}(\mathbf{f}_1\mathbf{f}_{\bar{2}} - \mathbf{f}_{\bar{1}}\mathbf{f}_2), & D &= \sqrt{-1}(\mathbf{f}_1\mathbf{f}_{\bar{2}} + \mathbf{f}_{\bar{1}}\mathbf{f}_2). \end{aligned}$$

$\mathcal{G}$ is reduced to $\mathcal{G}_1 + \mathcal{G}_2$. $\mathcal{G}_1$ is generated by H, A and B, and $\mathcal{G}_2$ is generated by H', C and D. Each of them is isomorphic to $o(3)$. Let

$$(1\text{-}3\text{-}3) \qquad \begin{aligned} \omega &= \mathbf{f}_1\mathbf{f}_2 \\ \Delta^+ &= C\omega + C\mathbf{f}_{\bar{1}}\mathbf{f}_{\bar{2}}\omega \\ \Delta^- &= C\mathbf{f}_{\bar{1}}\omega + C\mathbf{f}_{\bar{2}}\omega. \end{aligned}$$

$\Delta = \Delta^+ \oplus \Delta^-$ is the simultaneous eigenspace for the actions of $\sqrt{-1}\mathbf{e}_1\mathbf{e}_3 = (\mathbf{f}_1\mathbf{f}_{\bar{1}} - \mathbf{f}_{\bar{1}}\mathbf{f}_1)$ and $\sqrt{-1}\mathbf{e}_2\mathbf{e}_4 = (\mathbf{f}_2\mathbf{f}_{\bar{2}} - \mathbf{f}_{\bar{2}}\mathbf{f}_2)$ of eigenvalue $+1$. We have the following irreducible representation of $\mathcal{G}_1$:

$$(1\text{-}3\text{-}4) \qquad \Delta^+(H) = \sqrt{-1}\sigma_3 \quad \Delta^+(A) = \sigma_1 \quad \Delta^+(B) = \sigma_2\,,$$

where σ_k's are Pauli's matrices

$$\sigma_1 = \sqrt{-1}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \sigma_3 = \sqrt{-1}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2. Structures on a membrane

2.1. Let $\widehat{P}^n(\boldsymbol{C})$ be the dual projective space of $P^n(\boldsymbol{C})$, the homogeneous coordinates of $P^n(\boldsymbol{C})$ are denoted as $[t_o, \cdots, t_n]$ and those of $\widehat{P}^n(\boldsymbol{C})$ are $[u_0, \cdots, u_n]$. We consider the following double fibering:

$$\Lambda = \left\{([t_0, \cdots, t_n], [u_o, \cdots, u_n]) \in P^n(\boldsymbol{C}) \times \widehat{P}^n(\boldsymbol{C}) : t \cdot u = \sum_{j=0}^{n} t_j u_j = 0\right\},$$

with the projections on each factor $\pi : \Lambda \to P^n(\boldsymbol{C})$, $\hat{\pi} : \Lambda \to \widehat{P}^n(\boldsymbol{C})$. We set $U_0 = P^n(\boldsymbol{C}) - \{t_0 = 0\}$ and $\widehat{U}_0 = \widehat{P}^n(\boldsymbol{C}) - \{u_0 = 0\}$. For a neighborhood V of 0 in $U_0 \cong \boldsymbol{C}^n$, we define the dual $\widehat{V}$ of V by

$$\widehat{V} = \hat{\pi}(\pi^{-1}(P^n(\boldsymbol{C}) - \overline{V})).$$

For example, the dual domain of $0 \in \boldsymbol{C}^n$ is $\widehat{U}_0 = \hat{\pi}(\pi^{-1}(P^n(\boldsymbol{C}) - [1, 0, \cdots, 0]))$, which we shall denote by $\widehat{\boldsymbol{C}}^n$. The inhomogeneous coordinates on $U_0 \cong \boldsymbol{C}^n$ are written as $(z_1, \cdots, z_n)$, $z_i = \frac{t_i}{t_0}, i = 1, \cdots, n$. We put

$$R = \{z \in \boldsymbol{C}^n; |z| \le 1\}, \quad B = \{z \in \boldsymbol{C}^n; |z| = 1\}.$$

Let $\widehat{R}$ be the dual domain of R. If we denote the dual inhomogeneous coordinate on $\widehat{U}_o \cong \widehat{\boldsymbol{C}}^n$ by $(w_1, \cdots, w_n)$, $w_i = \frac{u_i}{u_0}, i = 1, \cdots, n$, we find $\widehat{R} = \{w \in \widehat{\boldsymbol{C}}^n; |w| \le 1\}$. We put $\widehat{B} = \{w \in \widehat{\boldsymbol{C}}^n; |w| = 1\}$.

REMARK. If V is a strictly convex set with smooth boundary then the space of continuous functions on $\overline{V}$ that are holomorphic on V and that on $\overline{\widehat{V}}$ are in duality, [K-1]. We shall prove in Theorem 5.12 the analogous duality between the null space of the Dirac operator on R and that on $\hat{R}$; this was one of our motivations. To extend our results in Theorems 5.10 and 5.12 to the case of the pair of convex sets V and $\widehat{V}$ will be interesting.

There is a smooth bijection $v : \boldsymbol{C}^n \setminus \{0\} \to \widehat{\boldsymbol{C}}^n \setminus \{0\}$ given by

$$w = v(z) = -\frac{\overline{z}}{|z|^2}. \tag{2-1-1}$$

$z \to (z, v(z))$ is a section of the projection π over $\boldsymbol{C}^n - \{0\}$. We patch up $\boldsymbol{C}^n$ and $\widehat{\boldsymbol{C}}^n$ together by v to obtain a differentiable manifold

$$M = \boldsymbol{C}^n \bigsqcup\nolimits_v \widehat{\boldsymbol{C}}^n. \tag{2-1-2}$$

We call M *membrane* (associated to the complex structure in consideration). M is a manifold homeomorphic to S^{2n}. We shall regard $\boldsymbol{C}^n$ and $\widehat{\boldsymbol{C}}^n$ as subsets of M.

We take $\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}, i = 1, \cdots, n\}$ for the local basis of $T^{\boldsymbol{C}}\boldsymbol{C}^n$. We use the similar notations on $\widehat{\boldsymbol{C}}^n$; $\{\frac{\partial}{\partial w_i}, \frac{\partial}{\partial \overline{w}_i}, i = 1, \cdots, n\}$. The real tangent vectors are denoted as $\sum_{i=1}^n (u_i \frac{\partial}{\partial z_i} + \overline{u}_i \frac{\partial}{\partial \overline{z}_i})$, $u \in \boldsymbol{C}^n$, and the same description on $\widehat{\boldsymbol{C}}^n$. The tangent bundle $TM \mapsto M$ is obtained by patching up these vectors by the transition function v_*. The matrix representation of the differential $v_* : T^{\boldsymbol{C}}(\boldsymbol{C}^n - \{0\}) \to T^{\boldsymbol{C}}(\widehat{\boldsymbol{C}}^n - \{0\})$ of transition function v is given by

$$(v_*)_z = \frac{\partial(w, \overline{w})}{\partial(z, \overline{z})} = \frac{1}{|z|^2} \begin{pmatrix} \frac{\overline{z}_i \overline{z}_j}{|z|^2} & (-\delta_{ij} + \frac{\overline{z}_i z_j}{|z|^2}) \\ (-\delta_{ij} + \frac{z_i \overline{z}_j}{|z|^2}) & \frac{z_i z_j}{|z|^2} \end{pmatrix}. \tag{2-1-3}$$

This is the composition of matrices of dilation, inversion and the reflection with respect to the plane perpendicular to the radial vector $\begin{pmatrix} z \\ \overline{z} \end{pmatrix}$.

2.2. We shall endow M with a conformally flat metric.

Lemma 2.1. *Let $z \in \boldsymbol{C}^n - \{0\}$ and $w = v(z) \in \widehat{\boldsymbol{C}}^n - \{0\}$. We have:*

$$(v_*)_w^{-1} = (v_*)_z \tag{1}$$

$$(1 + |w|^2)^{-2} \sum_{i=1}^n dw_i \otimes d\overline{w}_i = (1 + |z|^2)^{-2} \sum_{i=1}^n dz_i \otimes d\overline{z}_i. \tag{2}$$

Lemma 2.1 enables us to define a metric on M by the formula:

$$g = \begin{cases} (1 + |z|^2)^{-2} \sum_{i=1}^n dz_i \otimes d\overline{z}_i & \text{on } \boldsymbol{C}^n \\ (1 + |w|^2)^{-2} \sum_{i=1}^n dw_i \otimes d\overline{w}_i & \text{on } \widehat{\boldsymbol{C}}^n. \end{cases} \tag{2-2-1}$$

g is a G-invariant metric.

Put

$$\begin{aligned} \partial_i &= (1 + |z|^2) \frac{\partial}{\partial z_i}, \quad \partial_{\overline{i}} = (1 + |z|^2) \frac{\partial}{\partial \overline{z}_i}, \qquad i = 1, \cdots, n, \text{ on } \boldsymbol{C}^n, \\ \hat{\partial}_i &= (1 + |w|^2) \frac{\partial}{\partial w_i}, \quad \hat{\partial}_{\overline{i}} = (1 + |w|^2) \frac{\partial}{\partial \overline{w}_i}, \qquad i = 1, \cdots, n, \text{ on } \widehat{\boldsymbol{C}}^n. \end{aligned} \tag{2-2-2}$$

$(\partial_i, \partial_{\overline{i}})$ are skew g-orthonormal and $(\partial_i(z), \partial_{\overline{i}}(z)) = (\hat{\partial}_i(w), \hat{\partial}_{\overline{i}}(w))\sigma(z)$ for $w = v(z)$, where

$$\sigma(z) = |z|^2 (v_*)_z. \tag{2-2-3}$$

The principal G-frame bundle with respect to the frame $(\partial_i, \partial_{\overline{i}})$ is defined by the transition function σ, which we shall denote by $F(M)$.

Let $\mathcal{G}_M = F(M) \times_{Ad} \mathcal{G}$ be the adjoint bundle to the principal G-bundle $F(M)$ with the fiber the Lie algebra $\mathcal{G}$, where the fiber product is taken with respect to the adjoint action of G on $\mathcal{G}$.

2.3. The Clifford bundle $C(TM)$ is the subbundle of $C(T^C M)$ with the fiber at z generated by $TM_z \subset T^C M_z$. Equivalently, $C(TM)$ is the subbundle of $C(T^C M)$ consisting of those elements that change the sign under the bar operation as in (1-1-2).

The bundles $\mathcal{G}_M = F(M) \times_{Ad} \mathcal{G}$, $Pin(M) = Pin(TM)$ and $Spin(M) = Spin(TM)$ are contained in $C(TM)$. Let γ_0 denote the Clifford multiplication of the radial vector $\mathbf{n} = \frac{1}{|z|}\sum(z_j\partial_j + \overline{z}_j\partial_{\overline{j}}) \in Pin^1(2n)$. Then the principal $Spin(2n)$-bundle $Spin(M) \to M$ is isomorphic to the bundle obtained by the identification

$$\begin{aligned} &\boldsymbol{C}^n \times Pin^0(2n) \ni (z,g) \\ &\qquad \longleftrightarrow (w = v(z),\ \hat{g} = \overline{\gamma_0 g} = -\overline{\gamma_0} g\,) \in \widehat{\boldsymbol{C}}^n \times Pin^1(2n). \end{aligned} \tag{2-3-1}$$

The transition function of $Spin(M)$ descends to that of $F(M)$ by the bundle homomorphism ρ that comes from the homomorphism $\rho : Pin(2n) \longrightarrow O(2n)$, $\rho(-\overline{\gamma_0}) = \nu_*$. Let Δ and $\Delta^{\pm}$ be the representations of $Spin(2n)$ described in 1.3. We can form the spinor bundle $S = Spin(M) \times_{Spin(2n)} \Delta$. The even and odd half spinor bundles $S^{\pm}$ are described as follows: the transition function of $S^{\pm}$ is given by $\Delta(-\overline{\gamma_0})$ on $\boldsymbol{C}^n \cap \widehat{\boldsymbol{C}}^n$, hence the representation space $\Delta^{\pm}$ changes each other on $\boldsymbol{C}^n$ and on $\widehat{\boldsymbol{C}}^n$. Thus an even spinor $\varphi \in S^+$, for example, is a pair of $\varphi \in \mathcal{E}^0(C^n \times \Delta^+)$ and $\hat{\varphi} \in \mathcal{E}^0(\widehat{C}^n \times \Delta^-)$ such that

$$\hat{\varphi} = \Delta(-\overline{\gamma_0})\overline{\varphi}. \tag{2-3-2}$$

Here $\overline{\varphi}$ represents the complex conjugate, not the bar operation in $C(TM)$. The convention of taking the complex conjugate will be needed when we deal with the Dirac operator in section 4 and in accordance with the notion of "changing the charge" in physics. A spinor is said to have even (resp. odd) chirality if it is a section of the even (resp.odd) half spinor bundle S^+ (resp. S^-).

It is convenient for a later use to have the matrix form of $\Delta(\gamma_0)$. We restrict ourselves to the case $n = 2$, then, with respect to the basis $\{\omega,\ \partial_{\overline{1}}\partial_{\overline{2}}\omega,\ \partial_{\overline{1}}\omega,\ \partial_{\overline{2}}\omega\,\}$ of Δ, (1-2-3), $\Delta(\gamma_0)$ has the following matrix representation

$$\Delta(\gamma_0) = \frac{1}{|z|}\begin{pmatrix} 0 & 0 & z_1 & z_2 \\ 0 & 0 & -\overline{z}_2 & \overline{z}_1 \\ \overline{z}_1 & -z_2 & 0 & 0 \\ \overline{z}_2 & z_1 & 0 & 0 \end{pmatrix}. \tag{2-3-3}$$

2.4. In this section we restrict ourselves to the case $n = 2$. The general case is also treated by the same principle. The tangent bundle of B is

$$TB = \left\{ \begin{pmatrix} u \\ \overline{u} \end{pmatrix} \in TM \ :\ \ < \begin{pmatrix} u \\ \overline{u} \end{pmatrix}, \mathbf{n} >= 0 \right\},$$

where $\mathbf{n}$ is the radial vector field. The Clifford bundle $C(TB)$ is the subbundle :

$$C(TB) = \left\{ g \in C(TM)|B : \begin{array}{l} g_a\mathbf{n} + (-1)^{\deg g_a - 1}\mathbf{n}g_a = 0 \\ \text{for each homogeneous component } g_a \text{ of } g \end{array} \right\}.$$

See (1-2-1).

The frame bundle $F(B)$ is the subbundle of $F(M)$ with the fiber reduced to $F(B)_z = G_{\mathbf{n}} \cong O(3)$, where $G_{\mathbf{n}} = \{A \in G : A\mathbf{n} = \pm\mathbf{n}\}$ as is noted in 1.2. Similarly the principal $Spin(3)$-bundle $Spin(B)$ is given by

(2-4-1) $$Spin(B) = \{g \in Spin(M) : g(z)\mathbf{n} = \mathbf{n}g(z) \text{ at } z \in B\},$$

see (1-3-1). We have $Spin(B) = F(B) \times_{O(3)} Spin(3)$.

$Spin(3)$ has the representation space $\Delta^{\pm}$ (subsection 1.3), and $\gamma_0|\Delta_+ : \Delta_+ \longrightarrow \Delta^-$ defines an isomorphism between the representations. The even and odd spinor bundles $S_B^{\pm}$ on B are obtained by the patching (2-3-2) restricted on B.

2.5. We keep the assumption $n = 2$. We shall consider the following vector fields $\mathbf{n}$, θ_0, ϵ, $\overline{\epsilon}$ on $M - \{0, \hat{0}\} = \boldsymbol{C}^2 \cap \widehat{\boldsymbol{C}}^2$:

(2-5-1) $$\mathbf{n} = \frac{1}{2}(\nu + \overline{\nu}) \qquad \theta_0 = \frac{1}{2\sqrt{-1}}(\nu - \overline{\nu})$$

where

$$\nu = \begin{cases} \frac{1}{|z|}(z_1\partial_1 + z_2\partial_2) & \text{on } \boldsymbol{C}^2 - \{0\} \\ -\frac{1}{|w|}(w_1\hat{\partial}_1 + w_2\hat{\partial}_2) & \text{on } \widehat{\boldsymbol{C}}^2 - \{\hat{0}\} \end{cases}$$

$$\overline{\nu} = \begin{cases} \frac{1}{|z|}(\overline{z}_1\partial_{\overline{1}} + \overline{z}_2\partial_{\overline{2}}) & \text{on } \boldsymbol{C}^2 - \{0\} \\ -\frac{1}{|w|}(\overline{w}_1\hat{\partial}_{\overline{1}} + \overline{w}_2\hat{\partial}_{\overline{2}}) & \text{on } \widehat{\boldsymbol{C}}^2 - \{\hat{0}\} \end{cases}$$

and

$$\epsilon = \begin{cases} \frac{1}{|z|}(-\overline{z}_2\partial_1 + \overline{z}_1\partial_2) & \text{on } \boldsymbol{C}^2 - \{0\} \\ \frac{1}{|w|}(-w_2\hat{\partial}_{\overline{1}} + w_1\hat{\partial}_{\overline{2}}) & \text{on } \widehat{\boldsymbol{C}}^2 - \{\hat{0}\} \end{cases}$$

$$\overline{\epsilon} = \begin{cases} \frac{1}{|z|}(-z_2\partial_{\overline{1}} + z_1\partial_{\overline{2}}) & \text{on } \boldsymbol{C}^2 - \{0\} \\ \frac{1}{|w|}(-\overline{w}_2\hat{\partial}_1 + \overline{w}_1\hat{\partial}_2) & \text{on } \widehat{\boldsymbol{C}}^2 - \{\hat{0}\} \end{cases}.$$

$\mathbf{n}$ is the radial vector field (2-3-3) and θ_0, ϵ, $\overline{\epsilon}$ are longitudinal vector fields, that is, they are tangent to $|z| =$ constant sphere. Put

(2-5-2) $$\theta_1 = \frac{1}{2}(\epsilon + \overline{\epsilon}), \qquad \theta_2 = \frac{1}{2\sqrt{-1}}(\epsilon - \overline{\epsilon}).$$

With respect to the metric g induced on B the vector fields $\sqrt{2}\theta_0$, $\sqrt{2}\theta_1$, $\sqrt{2}\theta_2$ form an orthonormal frame of TB.

The dual differential forms $\mathbf{n}^*$, θ_0^*, ϵ^*, $\overline{\epsilon}^*$ on $M-\{0,\ \hat{0}\}$ are described as follows:

$$\begin{aligned} \mathbf{n}^* &= (\nu^* + \overline{\nu}^*) \\ \theta_0^* &= \sqrt{-1}(\nu^* - \overline{\nu}^*), \end{aligned} \tag{2-5-3}$$

where

$$\nu^* = \begin{cases} ((1+|z|^2)|z|)^{-1}(\overline{z}_1 dz_1 + \overline{z}_2 dz_2) & \text{on } \boldsymbol{C}^2 - \{0\} \\ -((1+|w|^2)|w|)^{-1}(\overline{w}_1 dw_1 + \overline{w}_2 dw_2) & \text{on } \widehat{\boldsymbol{C}}^2 - \{\hat{0}\} \end{cases}$$

$$\epsilon^* = \begin{cases} ((1+|z|^2)|z|)^{-1}(z_1 dz_2 - z_2 dz_1) & \text{on } \boldsymbol{C}^2 - \{0\} \\ ((1+|w|^2)|w|)^{-1}(\overline{w}_1 d\overline{w}_2 - \overline{w}_2 d\overline{w}_1) & \text{on } \widehat{\boldsymbol{C}}^2 - \{\hat{0}\}. \end{cases}$$

Put

$$\theta_1^* = \epsilon^* + \overline{\epsilon}^* \qquad \theta_2^* = \sqrt{-1}(\epsilon^* - \overline{\epsilon}^*). \tag{2-5-4}$$

These are dual differential forms of θ_1 and θ_2 respectively. Aside from the above vector fields $\nu, \overline{\nu}, \epsilon, \overline{\epsilon}$ and their dual forms we shall use another quartet of vector fields that play a complementary role in the sequel. We shall describe these vector fields in the following:

$$\mu = \frac{1}{|z|}(z_2\partial_2 + \overline{z}_1\partial_{\overline{1}}) \quad \text{on} \quad \boldsymbol{C}^2 - \{0\}$$

$$\delta = \frac{1}{|z|}(\overline{z}_2\partial_{\overline{1}} - z_1\partial_2) \quad \text{on} \quad \boldsymbol{C}^2 - \{0\}.$$

The formulas on $\widehat{\boldsymbol{C}}^2 - \{\hat{0}\}$ are omitted.

Put

$$\tau_0 = \frac{1}{2\sqrt{-1}}(\mu - \overline{\mu}), \qquad \tau_1 = \frac{1}{2}(\delta + \overline{\delta}), \qquad \tau_2 = \frac{1}{2\sqrt{-1}}(\delta - \overline{\delta}). \tag{2-5-5}$$

The vector fields $\sqrt{2}\tau_0$, $\sqrt{2}\tau_1$, $\sqrt{2}\tau_2$ form another orthonormal frame of TB.

For instance the Laplace-Beltrami operator Δ_1 on B is given by

$$\begin{aligned} \Delta &= \tfrac{3}{4}(\mathbf{n} + \mathbf{n})^2 + \Delta_1 \\ \Delta_1 &= (\theta_0^2 + \theta_1^2 + \theta_2^2) = (\tau_0^2 + \tau_1^2 + \tau_2^2), \end{aligned} \tag{2-5-6}$$

where Δ is the Laplace operator with respect to the metric g.

REMARK. $Spin(3) \simeq SU(2,C)$ acts on $B \simeq S^3$ naturally. Let R_g (resp. L_g) be the induced action on the continuous functions on B: $R_g f(z) = f(z \cdot g)$ (resp. $L_g f(z) = f(g^{-1} \cdot z)$). Then the differential representations of R_g (resp. L_g) are given by

$$\begin{aligned} dR(\sigma_3) &= \theta_0 \quad \text{resp.} \quad dL(\sigma_3) = \tau_0 \\ dR(\sigma_2) &= \theta_1 \quad \text{resp.} \quad dL(\sigma_2) = \tau_1 \\ dR(\sigma_1) &= \theta_2 \quad \text{resp.} \quad dL(\sigma_1) = \tau_2, \end{aligned} \tag{2-5-7}$$

σ_i's being Pauli's matrix. Let C be the Casimir operator. Then

$$dR(C) = dL(C) = \Delta_1. \tag{2-5-8}$$

3. Levi-Civita connections on M and B

3.1. The Levi-Civita connection of the metric g is the torsion free connection on the principal frame bundle $F(M)$ that is compatible with g. We shall describe it by a gauge potential defined by the transition function σ of $F(M)$, (2-2-3).

We consider the $\mathcal{G}^c$−valued 1 form $\sigma(z)^{-1}(d\sigma)_z$. Applying the bar operation on $\sigma^{-1}d\sigma$, we find $\overline{(\sigma(z)^{-1}(d\sigma)_z)} = -(\sigma(z)^{-1}(d\sigma)_z)$, therefore $\sigma^{-1}d\sigma$ is a $\mathcal{G}$-valued 1-form, see (1-1-2) and the discussion that follows it.

LEMMA 3.1
$$\sigma(w)^{-1}(d\sigma)_w = \sigma(z)(d\sigma^{-1})_z,$$
for $w = \upsilon(z) \in \widehat{C}^n$, $z \in C^n$.

We define a $\mathcal{G}$-valued 1 form (gauge potential) on each local coordinate as follows:

$$\begin{aligned} \Gamma(z) &= \frac{|z|^2}{1+|z|^2}\,\sigma(z)^{-1} \cdot (d\sigma)_z \quad \text{for } z \in C^n \\ \widehat{\Gamma}(w) &= \frac{|w|^2}{1+|w|^2}\,\sigma(w)^{-1} \cdot (d\sigma)_w \quad \text{for } w \in \widehat{C}^n. \end{aligned} \tag{3-1-1}$$

Then we have a connection form on $F(M)$ defined by

$$\gamma = \begin{cases} f^{-1}(z)(df)_z + f^{-1}(z)\Gamma(z)f(z) & \text{for } f \in F(M)|C^n \\ \hat{f}^{-1}(w)(d\hat{f})_w + \hat{f}^{-1}(w)\hat{\Gamma}(w)\hat{f}(w) & \text{for } \hat{f} \in F(M)|\widehat{C}^n. \end{cases}$$

In fact, from Lemma 3.1 we have, for $z \in C^n$ and $w = \upsilon(z) \in \widehat{C}^n$,

$$(\sigma\Gamma\sigma^{-1} + \sigma \cdot d\sigma^{-1})(z) = \frac{1}{(1+|z|^2)}\sigma(z)(d\sigma^{-1})_z = \frac{|w|^2}{1+|w|^2}\sigma(w)^{-1}(d\sigma)_w = \hat{\Gamma}(w),$$

and the connection form γ on $F(M)$ is well defined. γ is nothing but the Levi-Civita connection for the metric g. The connection γ lifts to define a connection on $Spin(M)$, and by the representation Δ of $Spin(2n)$ it induces the connection $\Delta(\gamma)$ on the spinor bundle S.

The explicit representation of the connection $\Gamma(z)$ as a $\mathcal{G}$-valued differential form is obtained by a direct calculation from the matrix representation of $\sigma(z)$, (2-2-3) and (2-1-3). If we write it using the representation λ of (1-1-1) it becomes

$$\begin{aligned}\sigma(z)^{-1}(d\sigma)_z = \frac{1}{|z|^2}\Big(&\sum_i (z_i d\overline{z}_i - \overline{z}_i dz_i)\partial_i\partial_{\overline{i}} \\ &+ \sum_{i<j}(z_i d\overline{z}_j - \overline{z}_j dz_i)\partial_i\partial_{\overline{j}} + (z_i dz_j - z_j dz_i)\partial_i\partial_j \\ &+ (\overline{z}_i d\overline{z}_j - \overline{z}_j d\overline{z}_i)\partial_{\overline{i}}\partial_{\overline{j}} + (\overline{z}_i dz_j - z_j d\overline{z}_i)\partial_{\overline{i}}\partial_j\Big). \end{aligned} \tag{3-1-2}$$

Note that $\mathcal{G}^C$ is generated by $\partial_i\partial_{\overline{i}}$, $\partial_i\partial_j$, $\partial_i\partial_{\overline{j}}$, $\partial_{\overline{i}}\partial_j$, $\partial_{\overline{i}}\partial_{\overline{j}}$, $i<j$.

COROLLARY. *The curvature of γ is given on $\boldsymbol{C}^n$ by*

$$\begin{aligned}\Omega(\gamma) = \frac{2}{(1+|z|^2)^2}\Big\{&\sum_i dz_i \wedge d\overline{z}_i\, \partial_i\partial_{\overline{i}} \\ &+ \sum_{i<j} dz_i \wedge d\overline{z}_j\, \partial_i\partial_{\overline{j}} + \sum_{i<j} dz_{\overline{i}} \wedge dz_j\, \partial_{\overline{i}}\partial_j \\ &+ \sum_{i<j} dz_i \wedge dz_j\, \partial_i\partial_j + \sum_{i<j} d\overline{z}_i \wedge d\overline{z}_j\, \partial_{\overline{i}}\partial_{\overline{j}}\Big\} \end{aligned} \tag{3-1-3}$$

and the same formula on $\widehat{\boldsymbol{C}}^n$.

$\Omega(\gamma)$ is a $\mathcal{G}$-valued 2-form: $\Omega(\gamma) \in \mathcal{E}^2(\mathcal{G}_M)$. Now we suppose $n=2$. Then we have the following matrix representation of $\Delta(\gamma)$ relative to spinor basis. An analogous formula is also valid for general n.

LEMMA 3.2. (1) *With respect to the basis $\{\,\omega = \partial_1\partial_2,\ \partial_{\overline{1}}\partial_{\overline{2}}\omega\,\}$ of Δ^+, $\Delta^+(\gamma)$ has the following matrix representation*

$$\Delta^+(\gamma) = \frac{|z|}{2}\begin{pmatrix} \sqrt{-1}\theta_0^* & -2\epsilon^* \\ 2\overline{\epsilon}^* & -\sqrt{-1}\theta_0^* \end{pmatrix}.$$

(2) *With respect to the basis $\{\,\partial_{\overline{1}}\omega,\ \partial_{\overline{2}}\omega\,\}$ of Δ^-,*

$$\Delta^-(\gamma) = \frac{|z|}{2}\begin{pmatrix} \sqrt{-1}\tau_0^* & -2\delta^* \\ 2\overline{\delta}^* & -\sqrt{-1}\tau_0^* \end{pmatrix}.$$

PROOF. Rewrite (3-1-2) with respect to the Chevalley basis of $\mathcal{G}$, (1-3-2), it turns out that $\Gamma(z)$ has the form:

$$\Gamma(z) = \frac{|z|}{2}\left(\theta_0^* H - \tau_0^* H' - \theta_2^* A + \tau_2^* C + \theta_1^* B - \tau_1^* D\right).$$

The representation follows from (1-3-4).

3.2. As in section 2.4 we assume $n = 2$. The Levi-Civita connection on B is the connection form on the $O(3)$-principal frame bundle $F(B)$ that is torsion free and compatible with the metric g. It lifts to define a connection on $Spin(B)$. Here we shall give directly the Levi-Civita connection form on $Spin(B)$.

LEMMA 3.3. *As relations on differential forms on B we have:*

$$d\theta_0^* = 2\theta_1^* \wedge \theta_2^*, \quad d\theta_1^* = 2\theta_2^* \wedge \theta_0^*, \quad d\theta_2^* = 2\theta_0^* \wedge \theta_1^*.$$

PROOF. Use the following fact:

$$\mathbf{n}^* \wedge \theta_0^* = \mathbf{n}^* \wedge \theta_1^* = \mathbf{n}^* \wedge \theta_2^* = 0 \quad \text{on } B.$$

Th fiber at each $z \in B$ of the bundle $\mathcal{G}_B$ is by definition $\mathcal{G}_{\mathbf{z}} = Lie(G_{\mathbf{z}})$ and we have $\mathcal{G}_{\mathbf{z}} = \{X \in \mathcal{G} : X\mathbf{z} = 0\}$. Recall that $\mathcal{G}_B$ acts on TB by the expression λ of (1-1-1). $\mathcal{G}_B$ becomes a subbundle of $\mathcal{G}_M$.

PROPOSITION 3.4. (1) *$\mathcal{G}_B$ has the following frame*

$$\mathbf{i} = \theta_1\theta_2, \quad \mathbf{j} = \theta_2\theta_0, \quad \mathbf{k} = \theta_0\theta_1.$$

(2) *These are the canonical basis of $\mathcal{G}_B$ with the relations*

$$[\mathbf{i}, \mathbf{j}] = 2\mathbf{k}, \quad [\mathbf{j}, \mathbf{k}] = 2\mathbf{i}, \quad [\mathbf{k}, \mathbf{i}] = 2\mathbf{j}.$$

Hence $\mathcal{G}_B$ is the adjoint bundle $F(B) \times_{Ad} o(3)$.

PROOF. The action of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ on TB is via λ of (1-1-1). We must first show $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are sections of $\mathcal{G}_M$. The bar operation changes each θ_j to $-\theta_j$ so it changes $\theta_1\theta_2$ to $\theta_2\theta_1$, thus $\mathbf{i}$ is in $C(TM)$, hence in $\mathcal{G}_M$, similarly for the others. We must next verify the condition $\lambda(\mathbf{i})\mathbf{n} = 0$ etc.. The matrix representation of $\mathbf{i}$ is

$$\lambda(\mathbf{i}) = \lambda(\theta_1\theta_2) = 2\sqrt{-1}\begin{pmatrix} -q & r & 0 & 0 \\ \bar{r} & -p & 0 & 0 \\ 0 & 0 & q & -\bar{r} \\ 0 & 0 & -r & p \end{pmatrix}$$

with $p = |z_1|^2$, $q = |z_2|^2$ and $r = z_1\overline{z}_2$. Therefore

$$\lambda(\mathbf{i})\mathbf{n} = \lambda(\mathbf{i})\begin{pmatrix} z \\ \overline{z} \end{pmatrix} = \begin{pmatrix} -qz_1 + rz_2 \\ \overline{r}z_1 - pz_2 \\ q\overline{z}_1 - \overline{r}\overline{z}_2 \\ -r\overline{z}_1 + p\overline{z}_2 \end{pmatrix} = 0.$$

Similarly,

$$\lambda(\mathbf{j})\mathbf{n} = 0, \qquad \lambda(\mathbf{k})\mathbf{n} = 0.$$

Hence $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are sections of $\mathcal{G}_B$. The commutation relations are easy to verify.

Proposition 3.4 shows that $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ generate the Lie algebra $\mathcal{G}_B$ of infinitesimal transformations of $F(B)$, hence of $Spin(B)$.

Let

$$\Gamma_B = \frac{1}{2}(\theta_0^*\mathbf{i} + \theta_1^*\mathbf{j} + \theta_2^*\mathbf{k}). \tag{3-2-1}$$

Γ_B is a $o(3)$-valued 1-form on B.

PROPOSITION 3.5. *Γ_B defines a connection on the associated bundle $TB = F(B) \times_{O(3)} R^3$ by the formula*

$$\begin{aligned} \nabla(\textstyle\sum_i f_i\theta_i) &= \textstyle\sum(df_i \cdot \theta_i + f_i\Gamma_B\theta_i), \quad i = 0,1,2 \\ \Gamma_B\theta_i &= \textstyle\sum_j \omega_i^j\theta_j \end{aligned} \tag{3-2-2}$$

with

$$\omega_i^j = \theta_k^*, \quad \omega_j^i = -\omega_i^j \quad (i,j,k) = (0,1,2) \text{ cyclically.} \tag{3-2-3}$$

PROOF. That Γ_B defines a connection is obvious. We shall verify (3-2-3). $\nabla\theta_p = \Gamma_B\theta_p$ is a TB-valued 1-form and by the same calculation as in Proposition 3.4,

$$\nabla_{\theta_0}\theta_p = \frac{1}{2}\lambda(\mathbf{i})\theta_p = \begin{cases} 0 & \text{if } p = 0 \\ -\theta_2 & \text{if } p = 1 \\ \theta_1 & \text{if } p = 2 \end{cases}.$$

Similarly

$$\nabla_{\theta_1}\theta_p = \begin{cases} \theta_2 & \text{if } p = 0 \\ 0 & \text{if } p = 1 \\ -\theta_0 & \text{if } p = 2 \end{cases} \qquad \nabla_{\theta_2}\theta_p = \begin{cases} -\theta_1 & \text{if } p = 0 \\ \theta_0 & \text{if } p = 1 \\ 0 & \text{if } p = 2 \end{cases}.$$

Thus (3-2-3) is proved.

Lemma 3.3 yields that Γ_B is torsion free, hence is the connection form for Levi-Civita connection. The corresponding connection on $Spin(B)$ is given by

$$\gamma_B = g(z)^{-1}(dg)_z + g(z)^{-1}\Gamma_B(z)g(z) \quad \text{for } g \in Spin(B).$$

Lemma 3.3 yields also the following:

COROLLARY. *The curvature form of γ_B is given by*

$$\Omega(\gamma_B)_i^j = 3\theta_i^* \wedge \theta_j^*. \tag{3-2-4}$$

$\Omega(\gamma_B) \in \mathcal{E}^2(\mathcal{G}_B)$.

Now the spinor representation of $\mathcal{G}_B$ is given by $\Delta^+(\mathbf{i}) = -\sigma_3$, $\Delta^+(\mathbf{j}) = \sigma_2$ and $\Delta^+(\mathbf{k}) = \sigma_1$. From this spin representation and the definition of γ_B we have the following:

LEMMA 3.6. *The Levi-Civita connection on B has the spinor representation on Δ^+ :*

$$\Delta^+(\gamma_B) = \frac{1}{2}\begin{pmatrix} \sqrt{-1}\theta_0^* & -2\epsilon^* \\ 2\bar{\epsilon}^* & -\sqrt{-1}\theta_0^* \end{pmatrix}.$$

Lemmas 3.2 and 3.6 show that the boundary value of $\Delta^+(\gamma)$ is $\Delta^+(\gamma_B)$.

4. Dirac operators

4.1. Let $S = Spin(M) \times_{Spin(2n)} \Delta$ be the spinor bundle described in section 2. The Levi-Civita connection γ induces a connection $\Delta(\gamma)$ on S that has the formula in Lemma 3.2.

The Dirac operator is a first order differential operator $\mathcal{D} : \mathcal{E}^0(M,S) \to \mathcal{E}^0(M,S)$ defined as follows. Let ∇ be the covariant derivative $\nabla\varphi = d\varphi + \Delta(\gamma)\varphi$. There is a natural identification of the tangent bundle TM and cotangent bundle T^*M. Hence we have the bundle homomorphism $\mu : S \otimes T^*M \to S$ coming from the Clifford multiplication. The Dirac operator is the composition of these two maps: $\mathcal{D} = \mu \circ \nabla$. $\mathcal{D}$ changes the half spinor to each other:

$$\begin{aligned} D &= \mathcal{D}|S^+ : \mathcal{E}^0(M,S^+) \to \mathcal{E}^0(M,S^-), \\ D^\dagger &= \mathcal{D}|S^- : \mathcal{E}^0(M,S^-) \to \mathcal{E}^0(M,S^+). \end{aligned} \tag{4-1-1}$$

We note that $\mathcal{E}^0(\widehat{C}^n, S^+) = \mathcal{E}^0(\widehat{C}^n, \Delta^-)$ and $\mathcal{E}^0(\widehat{C}^n, S^-) = \mathcal{E}^0(\widehat{C}^n, \Delta^+)$, see 2.3.

We shall give an explicit expression of the Dirac operator in the coordinate system on $C^n \subset M$. We suppose $n = 2$ for the sake of simplicity. The same calculation goes well also in general case but will be more complicated. Remember that the expression in the coordinate system on $\widehat{C}^2 \subset M$ has the same formula as that on C^2 because the connection γ does so, see Lemma 3.2.

To apply Clifford multiplication μ we do the following identification:

$$(1+|z|^2)^{-1}dz_i \leftrightarrow \partial_{\bar{i}}, \qquad (1+|z|^2)^{-1}d\bar{z}_i \leftrightarrow \partial_i.$$

These laws for calculus, Lemma 3.2 and (2-3-3) yield the following

LEMMA 4.1.

$$\mu\cdot\Delta(\gamma) = \frac{3}{2}\begin{pmatrix} 0 & & -z_1 & -z_2 \\ & & \bar{z}_2 & -\bar{z}_1 \\ -\bar{z}_1 & z_2 & 0 & \\ -\bar{z}_2 & -z_1 & & \end{pmatrix} = -\frac{3}{2}|z|\gamma_0.$$

Now we have the following expression of Dirac operator:

PROPOSITION 4.2. *Dirac operator $\mathcal{D}$ has the following matrix representation*:

$$\mathcal{D} = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}$$

(1) *on* C^2,

$$D = \begin{pmatrix} (1+|z|^2)\dfrac{\partial}{\partial z_1} - \dfrac{3}{2}\bar{z}_1 & -(1+|z|^2)\dfrac{\partial}{\partial \bar{z}_2} + \dfrac{3}{2}z_2 \\ (1+|z|^2)\dfrac{\partial}{\partial z_2} - \dfrac{3}{2}\bar{z}_2 & (1+|z|^2)\dfrac{\partial}{\partial \bar{z}_1} - \dfrac{3}{2}z_1 \end{pmatrix}$$

$$D^\dagger = \begin{pmatrix} (1+|z|^2)\dfrac{\partial}{\partial \bar{z}_1} - \dfrac{3}{2}z_1 & (1+|z|^2)\dfrac{\partial}{\partial \bar{z}_2} - \dfrac{3}{2}z_2 \\ -(1+|z|^2)\dfrac{\partial}{\partial z_2} + \dfrac{3}{2}\bar{z}_2 & (1+|z|^2)\dfrac{\partial}{\partial z_1} - \dfrac{3}{2}\bar{z}_1 \end{pmatrix}$$

(2) *the same matrix representations written in the coordinates* $(w,\overline{w})$ *on* $\widehat{C}^2$.

We can verify by direct calculation the next proposition which says that the operators given by the above matrices define in fact differential operators on M.

PROPOSITION 4.3. *Let φ be an even (resp. odd) spinor on a domain in C^2 and let $\hat{\varphi} = \Delta(-\overline{(\gamma_0|S^+)})\overline{\varphi}$ be the corresponding spinor on $\widehat{C}^2$. Then we have*

$$D\hat{\varphi}(w) = \widehat{D\varphi}(z) \quad \textit{resp. } D^\dagger\hat{\psi}(w) = \widehat{D^\dagger\psi} \textit{ for } w = \upsilon(z).$$

$\sqrt{-1}D$ and $\sqrt{-1}D^\dagger$ are elliptic operators and are formally adjoint to each other.

REMARK. We saw that the representation obtained has the same formula in two local coordinates; this is explained as *CPT- theorem*; the transition function of M restricted to B, $w = -\bar{z}$, represents the change of parity on $B \cong S^3$ and the time change is given by $\log|w| = -\log|z|$. The change of charge is described by the change of spinors $\widehat{\varphi}(w) = -\overline{\gamma_0 \varphi}(z)$, (2-3-2).

PROPOSITION 4.4.

$$\mathcal{D} = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} = \gamma_0 \cdot \left\{ \begin{pmatrix} \begin{matrix} \nu & -\bar{\epsilon} \\ \epsilon & \bar{\nu} \end{matrix} & 0 \\ 0 & \begin{matrix} \mu & -\bar{\delta} \\ \delta & \bar{\mu} \end{matrix} \end{pmatrix} - \frac{3}{2}|z| I \right\}$$

on $M - \{0, \hat{0}\}$.

In fact, we have

$$(1+|z|^2)\frac{\partial}{\partial z_1} = \frac{1}{|z|}(\bar{z}_1 \nu - z_2 \epsilon) = \frac{1}{|z|}(\bar{z}_1 \bar{\mu} + \bar{z}_2 \bar{\delta}) \text{ etc. .}$$

Replacing the entries of matrix representation of $\mathcal{D}$ in Proposition 4.2 we have the assertion. Here we used the abbreviation γ_0 instead of $\Delta(\gamma_0)$.

4.2. Lichnerowicz's theorem says that there is no harmonic spinor defined on M:

$$\{\varphi \in \mathcal{E}^0(M, S) : \mathcal{D}\varphi = 0\} = 0.$$

On the other hand there are plenty of harmonic spinors on $\boldsymbol{C}^2$ or on $\widehat{\boldsymbol{C}}^2$ as we shall see in the following.

Let $\mathcal{N}(S^+)$ be the sheaf of zero mode spinors of even chirality :

$$\mathcal{N}(S^+) = \left\{ \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in S^+ : D\varphi = 0 \right\}.$$

We put $p = (1+|z|^2)^{-\frac{3}{2}} p_1$ where p_1 is the projection $p_1 : \varphi \to \varphi_1$, and $j = j_2(1+|z|^2)^{\frac{3}{2}}$ where j_2 is the inclusion $j_2 : \psi \to \binom{0}{\psi}$. Let $\mathcal{H}$ be the sheaf of harmonic functions and $\mathcal{O}$ be the sheaf of holomorphic functions. These are sheaves of linear spaces.

PROPOSITION 4.5. *The next sequence is exact:*

$$0 \longrightarrow \mathcal{O} \xrightarrow{j} \mathcal{N}(S^+) \xrightarrow{p} \mathcal{H} \longrightarrow 0. \tag{4-2-1}$$

In fact, from the above expression of D, we see that $\phi = p\varphi$ is a harmonic function. Conversely, let ϕ be a harmonic function on an open neighborhood U

of $x \in M$. Let $\alpha = -\frac{\partial\phi}{\partial z_2}d\bar{z}_1 + \frac{\partial\phi}{\partial z_1}d\bar{z}_2$. α is a $\bar{\partial}$-closed $(0,1)$-form on U. There is an open neighborhood $V \subset U$ and a solution ψ of $\bar{\partial}u = \alpha$ on V. Then $\varphi = \begin{pmatrix} \varphi_1 = (1+|z|^2)^{\frac{3}{2}}\phi \\ \varphi_2 = (1+|z|^2)^{\frac{3}{2}}\psi \end{pmatrix}$ satisfies $D\varphi = 0$. Thus $p\varphi = \phi$ and p is surjective. The kernel of p consists of $\left\{ \begin{pmatrix} 0 \\ \psi \end{pmatrix} \in \mathcal{N}(S^+);\ \bar{\partial}(1+|z|^2)^{-\frac{3}{2}}\psi = 0 \right\}$, that proves the proposition. Let $G \subset C^2$ be a strongly pseudo-convex domain with smooth boundary.

COROLLARY 4.6. *The next sequence of linear spaces is exact and splits:*

$$0 \longrightarrow \mathcal{O}(G) \xrightarrow{j} \mathcal{N}(S^+)(G) \xrightarrow{p} \mathcal{H}(G) \longrightarrow 0.$$

The exactness follows from the vanishing of cohomology: $H^1(G, \mathcal{O}) = 0$. Let ϕ be a harmonic function on G and define α as above. Hörmander's projection P for the $\bar{\partial}$-problem gives us a global solution $\psi = P\alpha$ of $\bar{\partial}\psi = \alpha$ on $\mathbf{C}^2$. Put $\varphi_2 = (1+|z|^2)^{\frac{3}{2}}\psi$ so that we have $p\varphi = \phi$.

4.3. Now we shall give a matrix form of the Dirac operator on the boundary B. Let d_B be the exterior differentiation on B:

$$d_B\varphi = \theta_0\varphi\theta_0^* + \theta_1\varphi\theta_1^* + \theta_2\varphi\theta_2^*.$$

The Dirac operator on B, $D_B : \mathcal{E}(S_B^+) \longrightarrow \mathcal{E}(S_B^-)$, is defined by

$$D_B = \mu \cdot \nabla_B = \mu \cdot (d_B + \Delta^+(\gamma_B)),$$

where we do the identification $\theta_i^* \longleftrightarrow 2\theta_i$ when we apply Clifford multiplication μ.

Since $\sqrt{2}\theta_i\,, i = 0,1,2$ are orthonormal basis for the metric g we can write also $D_B = 2\sum_i \theta_i(\nabla_B)_{\theta_i}$.

Another type of Dirac operator on B that does not change the chirality is often used ([Hi]). This operator is defined by

$$\mathcal{P}\varphi = \tau D_B\varphi = 2\tau\sum_{i=0}^{2}\theta_i\nabla_{\theta_i}\varphi$$

for $\varphi \in \mathcal{E}^0(S^+)$, where τ is the Clifford multiplication of $\sqrt{-1}\theta_0\theta_1\theta_2$. We call $\mathcal{P}$ Hamiltonian. It is easy to see $\tau = -\gamma_0|S^-$. The adjoint Dirac operator of D_B is defined by the same formula as above but operating on the odd spinors: $D_B^\dagger : \mathcal{E}(S^-) \longrightarrow \mathcal{E}(S^+)$. We have

$$\mathcal{P} = -(\gamma_0|S^-)D_B, \qquad \mathcal{P}(\gamma_0|S^-) = D_B^\dagger. \tag{4-3-1}$$

The Dirac operator is decomposed to its radial part and Hamiltonian on the equator.

PROPOSITION 4.7.
(1)
$$D = (\gamma_0|S^+)(\mathbf{n} - \mathcal{P}),$$
$$D^\dagger = (\mathbf{n} + \mathcal{P})(\gamma_0|S^-).$$
(2)
$$\mathcal{P} = \begin{pmatrix} -\sqrt{-1}\theta_0 & \bar{\epsilon} \\ -\epsilon & \sqrt{-1}\theta_0 \end{pmatrix} + \tfrac{3}{2}|z| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

PROOF. (1) $d = \mathbf{n}\mathbf{n}^* + d_B$, so we have

$$\begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} = \gamma_0 \cdot \mathbf{n} + \begin{pmatrix} 0 & D_B^\dagger \\ D_B & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \mathbf{n}(\gamma_0|S^-) + \mathcal{P}(\gamma_0|S^-) \\ (\gamma_0|S^+)\mathbf{n} - (\gamma_0|S^+)\mathcal{P} & 0 \end{pmatrix}.$$

(2) From Propositions 4.4 we know that

$$D = (\gamma_0|S^+)\left\{\begin{pmatrix} \nu & -\bar{\epsilon} \\ \epsilon & \bar{\nu} \end{pmatrix} - \frac{3}{2}|z|\, I\right\}$$

$$= (\gamma_0|S^+)\cdot \mathbf{n} - (\gamma_0|S^+)\left\{\begin{pmatrix} -\sqrt{-1}\theta_0 & \bar{\epsilon} \\ -\epsilon & \sqrt{-1}\theta_0 \end{pmatrix} + \frac{3}{2}|z| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\},$$

hence

$$\mathcal{P} = \begin{pmatrix} -\sqrt{-1}\theta_0 & \bar{\epsilon} \\ -\epsilon & \sqrt{-1}\theta_0 \end{pmatrix} + \frac{3}{2}|z| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad \text{(4-3-2)}$$

By (4-3-2) we extend $\mathcal{P}$ to $\boldsymbol{C}^2 - \{0\}$. Then the above decomposition

$$D = (\gamma_0|S^+)(\mathbf{n} - \mathcal{P}) \qquad \text{(4-3-3)}$$

holds also on $\boldsymbol{C}^2 - \{0\}$.

PROPOSITION 4.8. *For $w = \upsilon(z)$, $z \in B$,*

$$\mathcal{P}\hat{\varphi} = \widehat{\mathcal{P}\varphi}.$$

We omit the proof.

4.4. We have

$$(4\text{-}4\text{-}1) \qquad < D\hat{\varphi}, \hat{\psi} >=< D\varphi, \psi >, \qquad < \hat{\varphi}, D^{\dagger}\hat{\psi} >=< \varphi, D^{\dagger}\psi >$$

for $\varphi \in \mathcal{E}^0(S^+)$, $\psi \in \mathcal{E}^0(S^-)$. The following Stokes' formula is valid:

PROPOSITION 4.9. *For* φ, $\psi \in \mathcal{E}^0(R, S)$, *we have*

$$\int_R \frac{1}{(1+|z|^2)^4} \left(< D\varphi, \psi > + < \varphi, D^{\dagger}\psi >\right) dV = \frac{1}{8} \int_B < \gamma_0\varphi, \psi > d\sigma.$$

On B the volume element $d\sigma$ is written as $d\sigma = \theta_0^* \wedge \theta_1^* \wedge \theta_2^* = -2\sqrt{-1}\theta_0^* \wedge \epsilon^* \wedge \bar{\epsilon}^*$. As operators on $L^2(B, d\sigma)$, θ_0 is selfadjoint: $(\theta_0\varphi, \psi) = (\varphi, \theta_0\psi)$, and the adjoint of ϵ is $-\bar{\epsilon}$. Therefore the Hamiltonian $\mathcal{P}$ is selfadjoint.

REMARK. If φ is a zero mode spinor of D then each component of φ is a harmonic function multiplied by $(1+|z|^2)^{\frac{3}{2}}$, hence $|\varphi|^2(z) =< \varphi, \varphi > (z)$ is a subharmonic function multiplied by $(1+|z|^2)^3$ and the L^2-norm of φ on R is smaller than a constant multiple of that on the boundary B. The constant is an absolute constant. Similar assertion is also true for the derivatives of φ.

5. Eigenvectors of the operator $\mathcal{P}$ on B

5.1. **(a)** Being a self adjoint elliptic differential operator, the operator

$$\mathcal{P} : \mathcal{E}^0(B, S_B^+) \longrightarrow \mathcal{E}^0(B, S_B^+)$$

has a discrete spectrum with real eigenvalues. We shall look for the eigenvectors of $\mathcal{P}$ by using the same argument as that we encounter when we construct a highest weight representation of $SU(2)$ on the space of spherical harmonic functions [Hi, Gol, Sch]. Detailed calculi in **(a)** are found in [K-2].

We note first the commutation relations:

$$[\sqrt{-1}\theta_0, \epsilon] = -\frac{1+|z|^2}{|z|}\epsilon, \quad [\sqrt{-1}\theta_0, \bar{\epsilon}] = \frac{1+|z|^2}{|z|}\bar{\epsilon}, \quad [\epsilon, \bar{\epsilon}] = 2\sqrt{-1}\frac{1+|z|^2}{|z|}\theta_0.$$

In particular, on $B = \{|z| = 1\}$, we have the following commutation relations which are the same as those of $sl(2, C)$:

$$(5\text{-}1\text{-}1) \qquad \begin{aligned} &[\sqrt{-1}\theta_0, \epsilon] = -2\epsilon, \quad [\sqrt{-1}\theta_0, \bar{\epsilon}] = 2\bar{\epsilon}, \quad [\epsilon, \bar{\epsilon}] = 4\sqrt{-1}\theta_0. \\ &[\sqrt{-1}\tau_0, \delta] = -2\delta, \quad [\sqrt{-1}\tau_0, \bar{\delta}] = 2\bar{\delta}, \quad [\delta, \bar{\delta}] = 4\sqrt{-1}\tau_0. \end{aligned}$$

In the following, we shall use double indices $\alpha = (\alpha_1, \alpha_2)$, α_i's being non-negative integers, and the notation $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}$ for $z \in \mathbf{C}^2$. We shall write also $|\alpha| = \alpha_1 + \alpha_2$. A polynomial that satisfies the following homogenuity condition:

$$P(az_1, bz_2, b\bar{z}_1, a\bar{z}_2) = a^{\alpha_1} b^{\alpha_2} P(z_1, z_2, \bar{z}_1, \bar{z}_2)$$

is said to be of class (α). The set of polynomials of class (α) is denoted by S_α. Let $\mathcal{H}$ be the set of harmonic polynomials on $\mathbf{C}^2$ and put $\mathcal{H}_\alpha = \mathcal{H} \cap S_\alpha$. We have the decomposition $S_\alpha = \mathcal{H}_\alpha \oplus |z|^2 S_{\alpha-1}$, hence $\dim \mathcal{H}_\alpha = |\alpha|+1$. On B every polynomial is written as a sum of harmonic polynomials in $\mathcal{H}_\alpha$. Put, for a double index α and $0 \le q \le |\alpha|$,

$$h_\alpha^q(z) = \left(\frac{|z|}{1+|z|^2}\epsilon\right)^q z^\alpha = \left(-\bar{z}_2 \frac{\partial}{\partial z_1} + \bar{z}_1 \frac{\partial}{\partial z_2}\right)^q z^\alpha. \tag{5-1-2}$$

For each α the set $\{h_\alpha^q;\ q = 0, \cdots, |\alpha|\}$ forms a basis of $\mathcal{H}_\alpha$.

Let $\widehat{\mathcal{H}}^\alpha$ be the space of harmonic polynomials that satisfy the homogenuity condition:

$$P(az_1, az_2, b\bar{z}_1, b\bar{z}_2) = a^{\alpha_1} b^{\alpha_2} P(z_1, z_2, \bar{z}_1, \bar{z}_2).$$

A polynomial that satisfies this condition is said to be of type (α). Let

$$\hat{h}_q^\alpha(z) = \left(\frac{|z|}{1+|z|^2}\delta\right)^q (\bar{z}_1^{\alpha_1} z_2^{\alpha_2}) = \left(\bar{z}_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial z_2}\right)^q (\bar{z}_1^{\alpha_1} z_2^{\alpha_2}). \tag{5-1-3}$$

The set $\{\hat{h}_q^\alpha;\ q = 0, \cdots, |\alpha|\}$ gives a basis of $\widehat{\mathcal{H}}^\alpha$ for each α. The next relation holds:

$$\hat{h}_a^{(p,q)} = (-1)^a \frac{q!}{b!} h_{(a,b)}^p$$

for the pairs (p, q) and (a, b) such that $p + q = a + b$.

PROPOSITION 5.1. *The following relations hold on B:*

(1) $\sqrt{-1}\theta_0\, h_\alpha^q = (|\alpha| - 2q) h_\alpha^q$
(2) $\epsilon\, h_\alpha^q = 2h_\alpha^{q+1}$
(3) $\bar{\epsilon}\, h_\alpha^q = -2q(|\alpha| - q + 1) h_\alpha^{q-1}$.

Similarly,

(1) $\sqrt{-1}\tau_0\, \hat{h}_q^\alpha = (|\alpha| - 2q)\hat{h}_q^\alpha$
(2) $\delta\, \hat{h}_q^\alpha = 2\hat{h}_{q+1}^\alpha$
(3) $\bar{\delta}\, \hat{h}_q^\alpha = -2q(|\alpha| - q + 1)\hat{h}_{q-1}^\alpha$.

PROOF. From the commutation relation (5-1-1) we have

$$\sqrt{-1}\theta_0 \epsilon^q = \sqrt{-1}\epsilon^q \theta_0 - \frac{1+|z|^2}{|z|} q\epsilon^q = \sqrt{-1}\epsilon^q \theta_0 - 2q\epsilon^q.$$

This implies the first equality. Similarly for the others.

Recall the relation mentioned at the end of 2.5 between the vector fields $\theta_0\,\tau_0\,, \epsilon, \ldots$ and the right and left representations R_g, and L_g of $SU(2, \boldsymbol{C})$ on the space of functions on B. From the above proposition we conclude that the space of harmonic polynomials $\mathcal{H}$ (restricted on B) is decomposed by the right action R of $SU(2)$ into $\mathcal{H} = \sum_r \sum_{|\alpha|=r} \mathcal{H}_\alpha$. Each induced representation $R_\alpha = (R, \mathcal{H}_\alpha)$ is an irrreducible representation with the highest weight $\frac{|\alpha|}{2}$. The corresponding assertion for the left representation L_g: $\mathcal{H} = \sum_r \sum_{|\alpha|=r} (L, \widehat{\mathcal{H}}^\alpha)$ holds similarly. We have

$$\Delta_1|\mathcal{H}_\alpha = dR_\alpha(\boldsymbol{C}) = dL^\alpha(\boldsymbol{C}) = \frac{|\alpha|(|\alpha|+2)}{8} I$$

by (2-5-8).

PROPOSITION 5.2.

(1) $\quad \left(h^p_\alpha,\, h^q_\beta\right)_B = \delta_{p,q}\,\delta_{\alpha,\beta}\,\frac{\alpha!}{|\alpha|+1}\,\frac{p!}{(|\alpha|-p)!}$,

(2) $\quad \left(\hat h^\alpha_p,\, \hat h^\beta_q\right)_B = \delta_{p,q}\,\delta_{\alpha,\beta}\,\frac{\alpha!}{|\alpha|+1}\,\frac{p!}{(|\alpha|-p)!}$.

Proof follows from

$$\int_B |z_1^a z_2^b|^2 d\sigma = \frac{a!b!}{(a+b+1)!}\,.$$

(**b**) Now we shall proceed to the discussion of eigenspinors. Since $\mathcal{P}$ commutes with the Laplace-Beltrami operator Δ_1 on S^3 we may consider $\mathcal{P}$ restricted to the eigenspace of Δ_1, that is, on $\mathcal{H}|B$.

Put, for α, $0 \le q \le |\alpha|+1$,

$$\phi^q_\alpha(z) = \begin{pmatrix} q\, h^{q-1}_\alpha(z) \\ -\,h^q_\alpha(z) \end{pmatrix}, \qquad z \in B. \tag{5-1-4}$$

We shall use the following convention:

$$\phi^0_\alpha = \begin{pmatrix} 0 \\ -h^0_\alpha \end{pmatrix}, \qquad \phi^{|\alpha|+1}_\alpha = \begin{pmatrix} (|\alpha|+1)\, h^{|\alpha|}_\alpha \\ 0 \end{pmatrix}.$$

From Propositions 4.7 and 5.1 we have

$$\mathcal{P}\phi^q_\alpha(z) = \left(|\alpha| + \frac{3}{2}\right)\phi^q_\alpha(z). \tag{5-1-5}$$

Therefore the numbers $r+\frac{3}{2}$, $r \geq 0$, are eigenvalues of $\mathcal{P}$. As for the multiplicity, the dimension of the linear space spanned by ϕ^q_α, $0 \leq q \leq r+1$, for each α with $|\alpha| = r$, is equal to $r+2$. The number of ways of choosing α is equal to $r+1$. Thus we have shown that the positive eigenvalues of $\mathcal{P}$ are $r+\frac{3}{2}$, $r = 0, 1, \cdots$, with the multiplicity $(r+1)(r+2)$.

Next we are going to discuss the negative eigenvalues. We define the following spinor on B:

$$\pi^\alpha_q(z) = \begin{pmatrix} \hat{h}^{\alpha_1,\alpha_2+1}_q(z) \\ \hat{h}^{\alpha_1+1,\alpha_2}_q(z) \end{pmatrix}. \tag{5-1-6}$$

We can verify immediately

$$\begin{pmatrix} -\sqrt{-1}\theta_0 & \bar{\epsilon} \\ -\epsilon & \sqrt{-1}\theta_0 \end{pmatrix} \pi^\alpha_0 = -(|\alpha| + 3)\pi^\alpha_0,$$

hence π^α_0 is an eigenvector of $\mathcal{P}$ that belongs to the eigenvalue $-(|\alpha| + \frac{3}{2})$. Since δ commutes with θ_0, ϵ and $\bar{\epsilon}$, all the

$$\pi^\alpha_q = \delta^q \pi^\alpha_0\,; \quad q = 0, 1, \cdots, r+1$$

are eigenvectors of $\mathcal{P}$ with the same eigenvalue $-(|\alpha| + \frac{3}{2})$.

Thus we have shown that, for each r, α such that $|\alpha| = r$ and q ranging from 0 to $r+1$, π^α_q is an eigenvector of $\mathcal{P}$ with the eigenvalue $-(r+\frac{3}{2})$. The multiplicity is $(r+1)(r+2)$. Since the spinors $\begin{pmatrix} h^p_\alpha \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ h^p_\alpha \end{pmatrix}$ are written by a linear combination of ϕ^p_α and $\pi^{(p,|\alpha|-p)}_{\alpha_1}$ the system obtained is complete. We have proved the following:

THEOREM 5.3. *The eigenvalues of $\mathcal{P}$ are*

$$\pm\left(\frac{3}{2} + r\right); \quad r = 0, 1, 2, \cdots$$

with multiplicity $(r+1)(r+2)$; in particular, there is no zero mode spinor of $\mathcal{P}$ and the spectrum is symmetric relative to 0.

PROPOSITION 5.4.

$$(\phi^q_\alpha, \phi^q_\alpha) = \frac{q!}{(|\alpha| + 1 - q)!}\alpha!$$

$$(\pi^\alpha_q, \pi^\alpha_q) = \frac{q!}{(|\alpha| + 1 - q)!}\alpha!$$

$$(\phi^p_\alpha, \pi^\beta_q) = 0.$$

The first and the second follow from Proposition 5.2. The third follows from: $[\bar{\epsilon},\delta]=[\epsilon,\bar{\delta}]=0$. We note that the corresponding expression of π_q^α on $\hat{B}=\{w\in\hat{C}^2;\,|w|=1\}$ becomes

$$\hat{\pi}_q^\alpha(w)=-\overline{(\gamma_0|S^+)\pi_q^\alpha(v^{-1}w)}=(-1)^{|\alpha|}\begin{pmatrix} q\hat{h}_{q-1}^\alpha(w)\\ -\hat{h}_q^\alpha(w)\end{pmatrix}, \tag{5-1-7}$$

a formula parallel to (5-1-4).

5.2. Extension of spinors from the equator

(**a**) In the following we shall discuss various extension problems of spinors from B to either of hemispheres as zero-mode spinors of Dirac operator D or $D^\dagger$.

LEMMA 5.5. *Let a be a smooth function on $(0,\infty)$ and let $f(z,\bar{z})=a(|z|)$. Then $\theta_0 f$, ϵf and $\bar{\epsilon}f$ vanish for $z\neq 0$ and*

$$\mathbf{n}f(z,\bar{z})=\frac{1+|z|^2}{2}a'(|z|).$$

LEMMA 5.6. *Let φ be an even spinor on B and extend it to a neighborhood of B by the formula*

$$\Phi(z)=\varphi\left(\frac{z}{|z|},\frac{\bar{z}}{|z|}\right).$$

Then $\mathbf{n}\Phi$ vanishes identically and

$$\mathcal{P}\Phi(z)=\frac{1+|z|^2}{2|z|}\left(\mathcal{P}\varphi-\frac{3}{2}\varphi\right)\left(\frac{z}{|z|}\right)+\frac{3}{2}|z|\varphi\left(\frac{z}{|z|}\right).$$

Let H be the space of square integrable spinors of even chirality on B. Let $H_\pm$ be the closed subspace of H spanned by the eigenvectors $\{\phi_\alpha^p\}_{p,\alpha}$ (resp. $\{\pi_p^\alpha\}_{p,\alpha}$) corresponding to the positive (resp. negative) eigenvalues $\lambda=\pm(|\alpha|+\frac{3}{2})$ of $\mathcal{P}$. In the sequel we denote by ϕ_λ either of eigenvectors $\{\phi_\alpha^p\}_{p,\alpha}$, $\{\pi_p^\alpha\}_{p,\alpha}$.

Let $H^s(R,S^\pm)$ denote the Sobolev space of spinors of even (resp. odd) chirality on $R=\{|z|\le 1\}$ with derivatives up to order s in L^2, similarly for $H^s(B,S^\pm)$. Let b be the trace to the boundary: $b: H^s(R,S^\pm)\longrightarrow H^{s-\frac{1}{2}}(B,S^\pm)$, $s>\frac{1}{2}$. b has a right inverse of Poisson-Szegö type integral: $K: H^{s-\frac{1}{2}}(B,S^\pm)\longrightarrow H^s(R,S^\pm)$, [Hö, T]. Since $\mathcal{P}$ is an elliptic operator of first order on B, the Sobolev norm of $\varphi=\sum_\lambda a_\lambda\phi_\lambda\in H^{\frac{1}{2}}(B,S^+)$ is given by $||\varphi||_{\frac{1}{2}}^2=\sum_\lambda(1+|\lambda|)|a_\lambda|^2$.

We put

$$\mathcal{N}(R)=\left\{f\in H^1(R,S^+): Df=0 \text{ on } R^\circ\right\}. \tag{5-2-1}$$

$f \in \mathcal{N}(R)$ is C^∞ on R°.

Let H^* be the space of square integrable spinors of odd chirality on B. γ_0 switches H and H^*: $(\gamma_0|S^+)H = H^*$, $(\gamma_0|S^-)H^* = H$. We shall define $H^*_\pm = (\gamma_0|S^+)H_\pm$. Let $H^1(R, S^-)$ be the Sobolev space of odd spinors on R and put

$$\mathcal{N}^\dagger(R)_0 = \left\{ f \in H^1(R, S^-) : \ D^\dagger f = 0 \text{ on } R^\circ \text{ and } f(0) = 0 \right\}. \tag{5-2-2}$$

On another hemisphere $\widehat{R}$ we have the boundary trace $\hat{b} : H^1(\widehat{R}, S^\pm) \longrightarrow H^{\frac{1}{2}}(B, S^\pm)$ which comes from the boundary trace $H^1(\widehat{R}, S^\pm) \longrightarrow H^{\frac{1}{2}}(\widehat{B}, S^\pm)$ followed by the patching up: $H^{\frac{1}{2}}(\widehat{B}, S^\pm) \ni \hat{\varphi} = -\overline{\gamma_0 \varphi} \longrightarrow \varphi \in H^{\frac{1}{2}}(B, S^\pm)$.

We put

$$\mathcal{N}(\widehat{R}) = \left\{ f \in H^1(\widehat{R}, S^+) : \ Df = 0 \text{ on } \widehat{R}^\circ \right\}. \tag{5-2-3}$$

Similarly,

$$\mathcal{N}^\dagger(\widehat{R})_0 = \left\{ f \in H^1(\widehat{R}, S^-) : \ D^\dagger f = 0 \text{ on } \widehat{R}^\circ \text{ and } f(\hat{0}) = 0 \right\}. \tag{5-2-4}$$

Let

$$r_\lambda(t) = t^{\lambda - \frac{3}{2}} \left(\frac{1+t^2}{2} \right)^{\frac{3}{2}}. \tag{5-2-5}$$

For an eigenspinor ϕ_λ of the Hamiltonian $\mathcal{P}$ with the eigenvalue λ, we put

$$\begin{aligned} \Phi_\lambda(z) &= r_\lambda(|z|)\phi_\lambda(\tfrac{z}{|z|}) \quad \text{on } \boldsymbol{C}^2 - \{0\}, \\ \widehat{\Phi}_\lambda(w) &= -\overline{\gamma_0 \Phi_\lambda(z)}|_{z=v^{-1}(w)} \quad \text{on } \widehat{\boldsymbol{C}}^2 - \{\hat{0}\}. \end{aligned} \tag{5-2-6}$$

These spinors patch together to define a spinor on $M - \{0, \hat{0}\}$. On $\widehat{\boldsymbol{C}}^2 - \{\hat{0}\}$, $\widehat{\Phi}_\lambda(w)$ has the form:

$$\widehat{\Phi}_\lambda(w) = |w|^{-\lambda - \frac{3}{2}} \left(\frac{1+|w|^2}{2} \right)^{\frac{3}{2}} \hat{\phi}_\lambda \left(\frac{w}{|w|} \right). \tag{5-2-7}$$

If λ is positive it is written as $\lambda = r + \frac{3}{2}$ and $\phi_\lambda = \phi^p_\alpha$ for some α with $|\alpha| = r$ and $0 \le p \le r+1$. Since ϕ^p_α is homogeneous of order r we have $\Phi_\lambda(z) = (\frac{1+|z|^2}{2})^{\frac{3}{2}} \phi^p_\alpha(z)$, which is actually defined on $\boldsymbol{C}^2$. On the other hand, for a negative λ we have $\lambda = -(r + \frac{3}{2})$ and $\phi_\lambda = \pi^\alpha_q$. Then, $\hat{\pi}^\alpha_q$ being homogeneous of order $|\alpha| = r$ from (5-1-7), $\widehat{\Phi}_\lambda(w) = (\frac{1+|w|^2}{2})^{\frac{3}{2}} \hat{\pi}^\alpha_q(w)$ is defined on $\widehat{\boldsymbol{C}}^2$.

LEMMA 5.7. (1) *For a positive eigenvalue $\lambda, \Phi_\lambda(z)$ is a zero-mode spinor of D on $\boldsymbol{C}^2$.*

(2) *For a negative eigenvalue $\lambda, \widehat{\Phi}_\lambda(w)$ is a zero-mode spinor of D on $\widehat{\boldsymbol{C}}^2$.*

PROOF. First we suppose $\lambda = r + \frac{3}{2}$, $r \geq 0$. From Lemmas 5.5 and 5.6 we have:

$$\begin{aligned} D\Phi_\lambda(z) &= \gamma_0(\mathbf{n} - \mathcal{P})\Phi_\lambda(z) \\ &= \frac{\gamma_0}{2}\left((1+|z|^2)r'_\lambda(|z|) - (\lambda - \frac{3}{2})\frac{1+|z|^2}{|z|}r_\lambda(|z|) - 3|z|r_\lambda(|z|)\right)\phi_\lambda\left(\frac{z}{|z|}\right). \end{aligned}$$

But $r_\lambda(t)$ satisfies the equation

$$(1+t^2)r'_\lambda(t) - \left(\lambda - \frac{3}{2}\right)\frac{1+t^2}{t}r_\lambda(t) - 3tr_\lambda(t) = 0.$$

Therefore $D\Phi_\lambda = 0$. The second assertion is proved similarly by virtue of Propositions 4.3 and 4.8.

Now we shall look at the space of odd spinors. For an eigenvalue λ we put $\psi^*_\lambda = (\gamma_0|S^+)\phi_\lambda$. $\psi^*_\lambda \in H^*$ and $\phi_\lambda = (\gamma_0|S^-)\psi^*_\lambda$. We define

$$\text{(5-2-8)} \qquad \Psi^*_\lambda(z) = (\gamma_0|S^+)\left(s_\lambda(|z|)\phi_\lambda\left(\frac{z}{|z|}\right)\right) = s_\lambda(|z|)\psi^*_\lambda\left(\frac{z}{|z|}\right),$$

where $s_\lambda(t) = (r_\lambda(t))^{-1} = t^{-(\lambda-\frac{3}{2})}(\frac{2}{1+t^2})^{\frac{3}{2}}$. If λ is negative we see as in Lemma 5.7 that Ψ^*_λ extends ψ^*_λ to $\boldsymbol{C}^2$ and $\Psi^*_\lambda(0) = 0$. Writing the equation satisfied by $s_\lambda(t)$ we can verify that

$$\begin{aligned} D^\dagger\Psi^*_\lambda &= (\mathbf{n} + \mathcal{P})\gamma_0\Psi^*_\lambda \\ &= \frac{1}{2}\left((1+|z|^2)s'_\lambda(|z|) + (\lambda - \frac{3}{2})\frac{1+|z|^2}{|z|}s_\lambda(|z|) + 3|z|s_\lambda(|z|)\right)\phi_\lambda\left(\frac{z}{|z|}\right) = 0. \end{aligned}$$

Then, as in Lemma 5.7, we have the following:

LEMMA 5.8. *Let*

$$\text{(5-2-9)} \qquad \begin{aligned} \Psi^*_\lambda(z) &= s_\lambda(|z|)\psi^*_\lambda(\tfrac{z}{|z|}) \\ \widehat{\Psi}^*_\lambda(w) &= -\overline{\gamma_0\,\Psi^*_\lambda(z)}|_{z=v^{-1}(w)} = |w|^{\lambda+\frac{3}{2}}(\tfrac{2}{1+|w|^2})^{\frac{3}{2}}\widehat{\psi}^*_\lambda(\tfrac{w}{|w|}). \end{aligned}$$

These patch together to define an odd spinor on $M - \{0, \hat{0}\}$ *which extends* ψ^*_λ.

(1) *For a positive eigenvalue* $\lambda, \widehat{\Psi}^*_\lambda(w)$ *is a zero-mode spinor of* $D^\dagger$ *on* $\widehat{\boldsymbol{C}}^2$ *that vanishes at* $\hat{0}$.

(2) *For a negative eigenvalue* $\lambda, \Psi^*_\lambda(z)$ *is a zero-mode spinor of* $D^\dagger$ *on* $\boldsymbol{C}^2$ *that vanishes at* 0.

LEMMA 5.9.

$$b : \mathcal{N}(R) \longrightarrow H_+ \cap H^{\frac{1}{2}}(B, S^+)$$
$$b : \mathcal{N}^\dagger(R)_0 \longrightarrow H_-^* \cap H^{\frac{1}{2}}(B, S^-)$$

are surjective.

In fact let $\varphi \in H_+ \cap H^{\frac{1}{2}}(B, S^+)$ and expand it in $\sum_{\lambda>0}(\varphi, \phi_\lambda)\phi_\lambda$. Put $K\varphi = \sum_{\lambda>0}(\varphi, \phi_\lambda)\Phi_\lambda(z)$. By virtue of the remark after Proposition 4.9 we can show that $K\varphi$ is well defined and is in $H^{\frac{1}{2}}(R, S^+)$. From Lemma 5.7 we have $DK\varphi = 0$. Let now $\psi^* \in H_-^* \cap H^{\frac{1}{2}}(B, S^-)$. Then $\psi^* = \sum_{\lambda<0}(\psi^*, \psi_\lambda^*)\psi_\lambda^*$. We note that $(\psi^*, \psi_\lambda^*) = ((\gamma_0|S^-)\psi^*, \phi_\lambda)$. Hence we can argue as above and see that $K^\dagger\psi^* = \sum_{\lambda<0}(\psi^*, \psi_\lambda^*)\Psi_\lambda^*$ is in $H^{\frac{1}{2}}(R, S^-)$ and $D^\dagger K^\dagger\psi^* = 0$. The fact that $K\varphi \in H^1(R, S^+)$ is verified in a routine way. For example, we refer to [Hö] or [T]. More generally, as a Poisson-Szegö type inverse of the trace b we can define

$$K\varphi(z) = \sum_{\lambda>0}(\varphi, \phi_\lambda)\Phi_\lambda(z) + \sum_{\lambda<0}\chi(|z|)(\varphi, \phi_\lambda)\Phi_\lambda(z) \tag{5-2-10}$$

for $\varphi \in H$, where $\chi(t)$ is a smooth function that vanishes near 0 and is equal to 1 for $t > \frac{1}{2}$. Similarly the boundary trace $b : H^1(R, S^-) \longrightarrow H^{\frac{1}{2}}(B, S^-)$ has the right inverse given by

$$K^\dagger\psi^*(z) = \sum_{\lambda>0}\chi(|z|)(\psi, \phi_\lambda)\Psi_\lambda^*(z) + \sum_{\lambda<0}(\psi, \phi_\lambda)\Psi_\lambda^*(z). \tag{5-2-11}$$

LEMMA 5.9-bis..

$$\hat{b} : \mathcal{N}^\dagger(\widehat{R})_0 \longrightarrow H_+^* \cap H^{\frac{1}{2}}(B, S^-)$$
$$\hat{b} : \mathcal{N}(\widehat{R}) \longrightarrow H_- \cap H^{\frac{1}{2}}(B, S^+)$$

are surjective.

THEOREM 5.10.

$$b : \mathcal{N}(R) \longrightarrow H_+ \cap H^{\frac{1}{2}}(B, S^+)$$
$$\hat{b} : \mathcal{N}(\widehat{R}) \longrightarrow H_- \cap H^{\frac{1}{2}}(B, S^+)$$

are isomorphic.

PROOF. Let $\Phi \in \mathcal{N}(R)$ and $\varphi = b\Phi = \phi_+ + \phi_-$ be the decomposition with $\phi_\pm \in H_\pm$. From lemma 5.9 ϕ_+ is extended to a spinor $\Phi_+ \in \mathcal{N}(R)$. The spinor $\Phi - \Phi_+$ on R has the boundary trace $\varphi - \phi_+ = \phi_-$ which is extended to define a spinor in $\mathcal{N}(\widehat{R})$ from Lemma 5.9-bis. Thus we have a spinor defined on the whole of M and annihilated by D, which must be 0. Hence $b\Phi \in H_+$. Similarly for the second assertion. We have known already the surjectivity of b and $\hat{b}$. The injectivity assertion follows from the last remark in subsection 4.4.

The following theorem is proved in the same way as for the previous one.

THEOREM 5.10-bis..

$$b : \mathcal{N}^\dagger(R)_0 \longrightarrow H^*_- \cap H^{\frac{1}{2}}(B, S^-)$$
$$\hat{b} : \mathcal{N}^\dagger(\widehat{R})_0 \longrightarrow H^*_+ \cap H^{\frac{1}{2}}(B, S^-)$$

are isomorphic.

COROLLARY 5.11.

$$\{\Psi \in H^1(R, S^-);\ D^\dagger\Psi = 0\} = \mathcal{N}^\dagger(R)_0\,,$$

that is, every zero mode spinor of odd chirality vanishes at 0.

In fact, put $\psi^* = b\Psi$ and let $\psi^* = \psi^*_+ + \psi^*_-$ be the decomposition to $\psi^*_\pm \in H^*_\pm$. By the theorem $\psi^*_- = b\Psi_-$ for a $\Psi_- \in \mathcal{N}^\dagger(R)_0$. Hence $\psi^*_+ = \psi^* - \psi^*_- = b(\Psi - \Psi_-)$ with $(\Psi - \Psi_-) \in H^1(R, S^-)$ and $D^\dagger(\Psi - \Psi_-) = 0$ on R°. On the other hand $\psi^*_+ = b\Lambda$ for a $\Lambda \in \mathcal{N}^\dagger(\widehat{R})_0$. Thus ψ^*_+ is the trace on B of a odd spinor on M that is annihilated by $D^\dagger$. It must be 0 and $\Psi = \Psi_- \in \mathcal{N}^\dagger(R)_0$.

(**b**) We define a pairing of H and H^* by

$$(\psi^* \,|\, \phi) = \int_B < \phi, \gamma_0\psi^* > \sigma(dz) \quad \text{for } \phi \in H \text{ and } \psi^* \in H^*. \tag{5-2-12}$$

Theorem 5.10 and Stokes' theorem (Proposition 4.9) yield that $H_\pm$ and $H^*_\mp$ are annihilated mutually by this pairing. On the other hand, $H_\pm$ and $H^*_\pm$ are respectively in duality. This is proved by Hahn-Banach's extension theorem.

A coupling between $\mathcal{N}(R)$ and $\mathcal{N}^\dagger(\widehat{R})_0$ is defined by

$$-\int_B \Phi(z) \cdot \widehat{\Psi}^*(v(z))\sigma(dz) = \int_B < \Phi, \gamma_0\Psi^* > \sigma(dz), \tag{5-2-13}$$

for $\Phi \in \mathcal{N}(R)$ and $\widehat{\Psi}^* \in \mathcal{N}^\dagger(\widehat{R})_0$, where $\widehat{\Psi}^*(w) = -\overline{\gamma_0\Psi^*(z)}|_{z=v^{-1}(w)}$. Also the coupling of $\widehat{\Phi} \in \mathcal{N}(\widehat{R})$ and $\Psi^* \in \mathcal{N}^\dagger(R)_0$ is defined by the integral:

$$-\int_B \Psi^*(z) \cdot \widehat{\Phi}(v(z))\sigma(dz) = \int_B < \Psi^*, \gamma_0\Phi > \sigma(dz). \tag{5-2-14}$$

The duality between $H_\pm$ and $H^*_\pm$ above and Theorem 5.10 prove the following:

THEOREM 5.12. (1) *The dual of* $\mathcal{N}(R)$ *is isomorphic to* $\mathcal{N}^\dagger(\widehat{R})_0$.
(2) *The dual of* $\mathcal{N}(\widehat{R})$ *is isomorphic to* $\mathcal{N}^\dagger(R)_0$.

6. Index of Dirac operator

The infinite dimensional Grassmannian on B associated to the eigenfunctions of $\mathcal{P}$ will serve as the set of lateral conditions of the Dirac operator on a hemisphere of S^4 .

6.1. H is always the space of square-integrable spinors of even chirality on B. $H_\pm$ is the subspace spanned by the eigenvectors ϕ^p_α (resp. π^α_p) corresponding to the eigenvalue $\pm(r+\frac{3}{2})$, where the indices are ranging over $0\le r$, $0\le p\le r+1$, $|\alpha|=r$. The orthogonal projections on $H_\pm$ are denoted by $Pr_{H_\pm}$.

The infinite dimensional Grassmannian $Gr(H)=Gr(H;\mathcal{P})$ consists of those closed subspaces W for which

(1) the orthogonal projection $Pr_{H_-}|W : W \longrightarrow H_-$ is a Fredholm operator,

(2) the orthogonal projection $Pr_{H_+}|W : W \longrightarrow H_+$ is a compact operator.

$Gr(H)$ is a Hilbert manifold modelled on the ideal of compact operators $H_- \longrightarrow H_+$, [P-S]. For a $W\in Gr(H)$, $Pr_{H_-}|W$ being a Fredholm operator, the dimensions of its kernel and cokernel are finite and we define the virtual dimension of W as the index of operator $Pr_{H_-}|W$; namely,

$$virtual\dim W = index(Pr_{H_-}|W) = \dim ker(Pr_{H_-}|W) - \dim coker(Pr_{H_-}|W).$$

We shall now define an ordered index set to enumerate the eigenvectors $\phi^p_\alpha, \pi^\beta_q$, which is useful for describing basic elements of our Grassmannian.

For a triplet $\lambda = \{\pm(r+\frac{3}{2});\ \alpha,\ p\}$, $0\le r$, $|\alpha|=r$, $0\le p\le r+1$, we put $-\lambda = \{\mp(r+\frac{3}{2});\ \hat{\alpha},\ r+1-p\}$, where $\hat{\alpha}=(\alpha_2,\alpha_1)$ for $\alpha=(\alpha_1,\alpha_2)$. Lexicographic order for the triplets $\lambda=\{s;\ \alpha,\ p\}$; $s=\pm(r+\frac{3}{2})$, is defined by $\lambda\ge\lambda'$ if either (i) $s>s'$, or (ii) $s=s'$, $\alpha_1>\alpha'_1$, or (iii) $s=s'$, $\alpha_1=\alpha'_1$ and $p\ge p'$. Hence $\lambda\ge\lambda'$ implies $-\lambda\le-\lambda'$. The smallest positive is $o_+ = (\frac{3}{2},(0,0),0)$ while the largest negative is $o_- = (-\frac{3}{2},(0,0),1)$.

We denote by $\mathcal{Z}$ the set of all triplets λ and $\mathcal{Z}_{\ge 0}$ (resp. $\mathcal{Z}_{<0}$) the set of all $\lambda\ge o_+$ (resp. $\lambda\le o_-$). We put also $\mathcal{Z}_{\le\alpha} = \{\beta\in\mathcal{Z};\ \beta\le\alpha\}$ for $\alpha\in\mathcal{Z}$.

A subset A of $\mathcal{Z}$ is called Maya diagram if both $A\cap\mathcal{Z}_{\ge0}$ and $A^c\cap\mathcal{Z}_{<0}$ are finite set. The integer $\chi(A)=\#(\mathcal{Z}_{\ge0}\cap A)-\#(\mathcal{Z}_{<0}\cap A^c)$ is called *charge* of A. $\mathcal{A}$ denotes the set of all Maya diagramms.For each Maya diagram $A\in\mathcal{A}$ with $\chi(A)=p$ there corresponds a unique increasing function $s:\mathcal{Z}_{<0}\to\mathcal{Z}$ such that (1) $s(\nu)=\nu+p$ for sufficiently small ν and (2) Image $(s)=A$.

For each $\pm\lambda = \pm(r+\frac{3}{2},\ \alpha,\ p)\in\mathcal{Z}$ we associate

$$\phi_\lambda = \begin{cases} \phi^p_\alpha & \text{if } \lambda\ge o_+ \\ \pi^\alpha_p & \text{if } \lambda\le o_- \end{cases}. \qquad \text{(6-1-1)}$$

(We shall often write, although somewhat confusing, $\lambda = \pm(r + \frac{3}{2})$.) For a Maya diagram A, H_A denotes the closed linear subspace of H spanned by $\{\phi_\lambda; \lambda \in A\}$. Evidently $H_{\mathbb{Z}_{<0}} = H_-$.

PROPOSITION 6.1. *For every $W \in Gr(H)$ there is a unique $A \in \mathcal{A}$ such that the orthogonal projection*

$$Pr_{H_A}|W : W \longrightarrow H_A$$

is an isomorphism.

The proof is similar to Proposition 7.1.6 of [P.S.].

Let $W \in Gr(H)$. From the above consideration there corresponds a unique Maya diagram $A = A_W$ and a linear map $w : H_A \longrightarrow H$ with the image $w(H_A) = W$ such that $w_- = Pr_{H_-} \cdot w$ is a Fredholm operator and $w_+ = Pr_{H_+} \cdot w$ is a compact operator. The index of this Fredholm operator is found to be equal to the charge $\chi(A) : \chi(A) = virtual \dim W$. W has the following particular frames called *canonical basis*:

$$w = (w_\mu)_{\mu \in A}$$

(6-1-2)
$$w_\mu = \phi_\mu + \sum_{A^c \ni \nu > \mu} u^\nu_\mu \phi_\nu .$$

Let $W^\perp$ be the orthogonal complement of W in H. $W^\perp$ has the frames

$$w^\perp = (w^\perp_\lambda)_{\lambda \in A^c}$$

(6-1-3)
$$w^\perp_\lambda = \phi_\lambda - \sum_{A \ni \sigma < \lambda} u^\lambda_\sigma \phi_\sigma .$$

(u^λ_μ) is a compact operator. The frames $\{w_\mu, w^\perp_\lambda\}_{\mu \in A, \lambda \in A^c}$ are uniquely determined by $W \in Gr(H)$.

6.2. For a $W \in Gr(H)$, we denote by Pr_W the projection operator of H onto W. $Pr_{W^\perp}$ denotes the projection on $W^\perp$. The space of all C^∞ spinors of even chirality on R satisfying $Pr_{W^\perp}(\varphi|B) = 0$ will be denoted by $C^\infty(R, S^+; W^\perp)$ and C^∞_c will denotes spinors vanishing for $|z|$ less than some $\epsilon > 0$. Similarly the space $C^\infty(R, S^-; W)$ is defined by the condition $Pr_W(\gamma_0 \psi|B) = 0$.

From Stokes' formula, Proposition 4.9, we have

(6-2-1)
$$(D\varphi, \psi) + (\varphi, D^\dagger \psi) = 0$$

if $\varphi \in C^\infty_c(R, S^+; W^\perp)$ and $\psi \in C^\infty_c(R, S^-; W)$.

PROPOSITION 6.2. *Let $W \in Gr(H)$. There is a linear operator*

$$Q : C_c^\infty(R, S^-) \longrightarrow C^\infty(R, S^+;\, W^\perp)$$

such that

(1) $DQ = Id$ *on* $C_c^\infty(R, S^-)$
(2) $QD = Id$ mod *a finite rank operator on* $C_c^\infty(R, S^+;\, W^\perp)$
(3) Q *extends to a continuous map* $L^2(R, S^-) \longrightarrow H^1(R, S^+; W^\perp)$.

.

PROOF. For a smooth function $g(t)$ on $[0, \infty)$ vanishing on a neighborhood of 0 we put

$$F_\lambda^0 g(u) = r_\lambda(u) \int_0^u s_\lambda(t) \frac{2}{1+t^2} g(t)dt$$
$$F_\lambda^1 g(u) = -r_\lambda(u) \int_u^1 s_\lambda(t) \frac{2}{1+t^2} g(t)dt.$$

$F_\lambda g$ $(= F_\lambda^0 f$ or $F_\lambda^1 g)$ is a smooth function on $[0,1]$ and solves the equation

$$(1+u^2)\frac{d}{du} f(u) - \left(\lambda - \frac{3}{2}\right) \frac{1+u^2}{u} f - 3uf(u) = 2g(u).$$

The same calculations as in Lemma 5.7 implies that

$$D(F_\lambda g)(|z|)\phi_\lambda \left(\frac{z}{|z|}\right) = \gamma_0 g(|z|)\phi_\lambda \left(\frac{z}{|z|}\right).$$

We note that $(F_\lambda g + c r_\lambda)(|z|)\phi_\lambda(\frac{z}{|z|})$ for any constant c satisfies also the same equation; this fact will be used later. We have $F_\lambda^1 g(1) = 0$. To prove the proposition first we shall treat the case $W = H_A$ for a Maya diagram A. Let $\psi \in C_c^\infty(R, S^-)$ and expand it in

$$\psi(z) = (\gamma_0 | S^+) \sum g_\lambda(|z|)\phi_\lambda \left(\frac{z}{|z|}\right).$$

For each g_λ we associate the above solution functions:

$$f_\lambda = \begin{cases} F_\lambda^0 g_\lambda & \text{for } \lambda \in A \\ F_\lambda^1 g_\lambda & \text{for } \lambda \in A^c \end{cases},$$

and put $Q_\lambda \psi(z) = f_\lambda(|z|)\phi_\lambda(\frac{z}{|z|})$. Then $Q_\lambda \psi(z) = 0$ on B for each $\lambda \in A^c$, that is, $Pr_{H_A^\perp}(bQ_\lambda \psi) = 0$. For each λ, Q_λ defines a map from $C_c^\infty(R, S^-)$ to $C^\infty(R, S^+;\, H_A^\perp)$. Formally $Q = \sum Q_\lambda$ satisfies (1) and (2). Now $A^c \cap Z_{<0}$ and

$A \cap \mathcal{Z}_{\geq 0}$ consist of finite elements. Except for these finite number of λ's we have the following estimates

$$\int_0^1 |f_\lambda(u)|^2\, du \leq C_1 \int_0^1 |g_\lambda(t)|^2 dt$$
$$\int_0^1 |\frac{d}{du} f_\lambda(u)|^2\, du \leq C_2 \int_0^1 |g_\lambda(t)|^2 dt\,.$$

Therefore the sum $Q\psi(z) = \sum_\lambda Q_\lambda \psi(z)$ converges for $|z| \leq 1$ and satisfies the desired properties. In the above we have referred to the argument of Proposition 2.5 in [A-P-S]. For a general $W \in Gr(H)$ we revise the argument as follows. We expand ψ by the canonical basis of (6-1-2) and (6-1-3);

$$\psi = (\gamma_0 | S^+) \left\{ \sum_{\mu \in A} h_\mu(|z|) w_\mu(\frac{z}{|z|}) + \sum_{\rho \in A^c} h_\rho(|z|) w_\rho^\perp(\frac{z}{|z|}) \right\}$$

and rewrite it with respect to the basis $\{\phi_\lambda\}$.

$$\psi(z) = (\gamma_0 | S^+) \left\{ \sum_{\mu \in A} g_\mu(|z|) \phi_\mu(\frac{z}{|z|}) + \sum_{\rho \in A^c} g_\rho(|z|) \phi_\rho(\frac{z}{|z|}) \right\}$$

with

$$g_\mu(t) = h_\mu(t) - \sum_{\mu < \nu \in A^c} h_\nu(t) u_\mu^\nu \quad \text{for } \mu \in A$$
$$g_\rho(t) = h_\rho(t) + \sum_{\rho > \sigma \in A} h_\sigma(t) u_\sigma^\rho \quad \text{for } \rho \in A^c.$$

Put

$$G_\mu(t) = F_\mu^0 h_\mu(t) - \sum_{\mu < \nu \in A^c} u_\mu^\nu F_\mu^1 h_\nu(t) \quad \text{for } \mu \in A$$
$$G_\rho(t) = F_\rho^1 h_\rho(t) + \sum_{\rho > \sigma \in A} u_\sigma^\rho \left(F_\rho^0 h_\sigma(t) + c\, r_\rho(t) \right) \quad \text{for } \rho \in A^c,$$

where $c = F_\sigma^0 h_\sigma(1) - F_\rho^0 h_\sigma(1)$. Then by the above argument the sum

$$Q\psi(z) = \sum_{\mu \in A \cup A^c} G_\mu(|z|) \phi_\mu(\frac{z}{|z|})$$

converges in $H^1(R, S^+)$ and satisfies (1) and (2). We can verify that, for $z \in B$,

$$Q\psi(z) = \sum_{\mu \in A} F_\mu^0 h_\mu(1) w_\mu(z) \in W.$$

The proposition is proved.

D has the closed extension $D_{W^\perp}$ with the domain

$$H^1(R, S^+, W^\perp) = \{\varphi \in H^1(R, S^+) : \; Pr_{W^\perp}(\varphi|B) = 0\}$$

and $D^\dagger$ has the closed extension $D^\dagger_W$ with the domain

$$H^1(R, S^-; W) = \{\psi \in H^1(R, S^-) : Pr_W(\gamma_0\psi|B) = 0\}\,.$$

PROPOSITION 6.3. *$\sqrt{-1}D^\dagger_W$ and $\sqrt{-1}D_{W^\perp}$ are adjoint to each other.*

The proof follows from the argument of [A-P-S, p.51]. Q in Proposition 6.2 gives a bounded inverse of D with domain $C^\infty(R, S^+;\, W^\perp)$ and similarly we get a bounded inverse R of $D^\dagger$ with the boundary condition $Pr_W(\gamma_0\psi|B) = 0$. Then $\sqrt{-1}R$ and $\sqrt{-1}Q$ are mutually adjoint from (6-2-1). Since adjoints commute with inverses, the proposition is established.

For $W = H_-$, in particular, we put $D_{W^\perp} = D_+$, $D^\dagger_W = D^\dagger_-$. Then $\sqrt{-1}D_+$ and $\sqrt{-1}D^\dagger_-$ are adjoint to each other.

It follows from the proposition that the domain of $D_{W^\perp}$ is the kernel of the composite maps $b : H^1(R, S^+) \longrightarrow H^{\frac{1}{2}}(B, S^+) \longrightarrow H$ and the projection $Pr_{W^\perp} : H \longrightarrow W^\perp$.

THEOREM 6.4. (1) $\dim ker D_{W^\perp} = \dim ker(Pr_{H_-}|W)$.
(2) $\dim ker D^\dagger_W = \dim coker(Pr_{H_-}|W)$.
(3) $index D_{W^\perp} = virtual \dim W$.

PROOF. Theorem 5.10 implies that

$$\begin{aligned} ker D_{W^\perp} &= \{\varphi \in \mathcal{N}(R) : \; Pr_{W^\perp}(\varphi|B) = 0\} \\ &\xrightarrow[b]{\simeq} \{\varphi' \in H_+ : \; Pr_{W^\perp}(\varphi') = 0\} = ker(Pr_{W^\perp}|H_+). \end{aligned}$$

But since $ker(Pr_{W^\perp}|H_+) = H_+ \cap W = ker(Pr_{H_-}|W)$ we have the isomorphism $ker D_{W^\perp} \simeq ker(Pr_{H_-}|W)$. From Corollary 5.11 we have

$$\begin{aligned} ker D^\dagger_W &= \{\Psi \in \mathcal{N}^\dagger(R)_0 ;\; Pr_W(\gamma_0\Psi|B) = 0\} \\ &\xrightarrow[b]{\simeq} \{\psi^* = \gamma_0\psi \in H^*_- ;\; Pr_W(\psi) = 0\} = ker(Pr_W|H_-)' \end{aligned}$$

which is the same subspace as $coker(Pr_{H_-}|W)$. Hence the boundary trace b gives the isomorphism $ker D^\dagger_W \simeq coker(Pr_{H_-}|W)$. (3) follows from (1), (2) and Proposition 6.3.

COROLLARY 6.5. (1) $ker D_+ = ker(Pr_{H_-}|H_-) = 0$.

(2) $kerD_-^\dagger = coker(Pr_{H_-}|H_-) = 0.$

In [A-B-P] and [S-2] it is shown that

$$\begin{aligned} IndexD_+ &= \dim kerD_+ - \dim kerD_-^\dagger \\ &= \int \widehat{A}(R) - \frac{h+\eta(0)}{2} \end{aligned} \tag{6-2-2}$$

where h is the dimension of the null space of $\mathcal{P}$ on B and $\eta(0)$ is the η-invariant of $\mathcal{P}$. By (6-2-2), Theorem 5.3 and Corollary 6.5 we have an alternative proof of

$$\int_R \widehat{A}(R) = 0. \tag{6-2-3}$$

6.3. Here we discuss briefly the index of the Dirac operator coupled with gauge potentials [A-S]. Let $\mathcal{A}$ be the set of gauge potentials on S^4 with values in $su(N)$ and $\mathcal{G}$ be the group of gauge transformations leaving the north pole fixed. Then the space $\mathcal{A}/\mathcal{G}$ has the same homotopy type as $\Omega^3(SU(N))$ and the map α giving the homotopy equivalence is described by the parallel transport by means of $A \in \mathcal{A}$ around closed curves parametrized by the equator S^3, [S-1]. Mickelsson [M] showed that $\Omega^3(SU(N))$ acts on $Gr_2(H^N)$, where $Gr_p(H^N) \subset Gr(H^N)$ is the infinite dimensional Grassmannian defined by the Schatten ideal L_{2p}, that is, in the second condition of the definition stated in 6.1 we demand $Pr_{H_+^N}|W$ to be in L_{2p}. Hence to every gauge potential $A \in \mathcal{A}$ we can associate

$$W_A = \alpha(A) \cdot H_-^N \in Gr(H^N).$$

The virtual dimension of W_A is seen to be equal to the index of the multiplication operator $\alpha(A)$ (which is defined by the Fredholm entry of the multiplication operator) and by a little modification of Atiyah-Janich theorem we can show that the index of the multiplication operator $\alpha(A)$ is equal to the degree of $\alpha(A)$ in $\Omega^3(SU(N))$. From the Atiyah-Singer index theorem the last number is equal to the index of the Dirac operator D_A on M coupled to gauge potential A. Hence we have

$$indexD_A = virtual \dim W_A.$$

It is desirable to make clear the correspondence of D_A and $D_{W_A^\perp}$ and to prove the above index formula from Theorem 6.4.

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