Spinor analysis on $\mathbf{C}^2$ and on conformally flat 4-manifolds

By Tosiaki Kori

Reprinted from
JAPANESE JOURNAL OF MATHEMATICS
Vol. 28, No. 1, pp. 1–30
October 2002

Japan. J. Math.
Vol. 28, No. 1, 2002

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By Tosiaki KORI[1]

(Received August 24, 2000)
(from Tokyo Journal of Mathematics)

Abstract. The present paper is concerned with extensions of complex analysis on the complex plane C to conformally flat 4-manifolds. We shall give in an explicit form a fundamental system of spinors that will serve as the basis vectors for the Laurent expansion. Restricted to a sphere around the center of the expansion these spinors form a complete orthonormal system of eigenspinors of the tangential Dirac operator on the sphere, and give a basis of the representations of Spin(4). We shall also give the definition of meromorphic spinors and residues, and prove under some hypothesis that, on a compact conformally flat 4-manifold, the sum of the residues of a meromorphic spinor is zero.

0. Introduction

The Dirac operator on the complex plane C is the Cauchy-Riemann operator $\overline{\partial}$ and the elementary complex function theory is the investigation of various properties of zero mode functions of $\overline{\partial}$ and their singularities, that is, properties of holomorphic functions and meromorphic functions. There the main theorems are the Cauchy integral formula, the Laurent expansion, the residue theorem and the Mittag-Leffler theorem etc. In this paper we shall extend these theorems to the Dirac operator $D : C^\infty(\mathrm{C}^2, S^+) \longrightarrow C^\infty(\mathrm{C}^2, S^-)$, where $S^\pm$ are the two spin bundles associated to two half-spin representations of Spin(4). An even spinor $\varphi \in C^\infty(\mathrm{C}^2, S^+)$ such that $D\varphi = 0$ is called a zero mode spinor. We shall prove the Cauchy integral formula and a Laurent expansion theorem for zero mode spinors. We shall give the definition of meromorphic spinors and residues and then prove the residue theorem. Similar ideas were investigated in quaternionic analysis by Fueter and his school in the 1930s [F], and by Sudbery [S]. More generally, extensions of complex analysis to R^n have been studied by many authors, starting with Dixon [Dx] and Moisil [Moi], and the Clifford analysis has been extensively developed by

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2000 *Mathematics Subject Classification.* 58G30, 53C21.
Key words. Dirac operator, Residue theorem, Conformally flat manifold.
[1]Research supported by Promotion for Sciences of the ministry of education in Japan (No. 09640134) and Waseda University Grant for special research project (No. 99A-166).

Delanghe and his school [D, B-D-S, D-S-Sč, VL], by Gilbert and Murray [G-M], and by McIntosh [Mc], Mitrea [Mit], Ryan [R], and Souček [Sč], among others. The analysis of Dirac operators and their solutions are investigated also on arbitrary Riemannian manifolds by Booss and Wijciechowski [B-W], Calderbank [C], Gürlebeck and Sprössig [G-Sp] and Mitrea [Mi]. We cite [B-D-S] and [D-S-Sč] as the standard references. But in this paper we need not rely on special knowledge from this genre. We present all our results by direct calculations and, as the reader will find, the arguments used here are well known ones of classical function theory. So our approach has the merit of being self -contained and perhaps well-adapted to the needs of mathematical physicists. Since the theory is covariant under conformal transformations concepts like the order of pole and the residue of a meromorphic spinor are invariant under conformal transformations on R^4, hence our theory can be extended to conformally flat manifolds.

Now we shall explain the contents of each section. Section 1 is preliminary. The half Dirac operator $D : C^{\infty}\left(\mathrm{C}^2, S^+\right) \longrightarrow C^{\infty}\left(\mathrm{C}^2, S^-\right)$ has the polar decomposition:

$$D = \gamma_+ \left(\frac{\partial}{\partial n} - \partial\!\!\!/ \right), \tag{0-1}$$

where γ_+ is the operator of chiral change, and $\partial\!\!\!/$ is the tangential Dirac operator. We shall introduce a basis of harmonic polynomials on R^4 that are related to highest weight representations of $SU(2)$. We then construct, using these harmonic polynomials as components, a fundamental class of zero mode spinors on $\mathrm{C}^2 \setminus \{0\} : \{\phi^{\pm(m,l,k)}\}$, that will serve as the basis vectors of the Laurent expansion. Restricted to the unit sphere $\simeq S^3$ these spinors form a complete orthonormal system of eigenspinors of the tangential Dirac operator. The system gives also the highest weight representations of Spin(4). Things are similar to the case of the Cauchy-Riemann operator $\frac{\partial}{\partial \bar{z}}$ on C, where $z^{\pm n}$ serve as the basis vectors of Laurent expansion and $e^{\pm in\theta}$ are the eigenfunctions of $-i\frac{\partial}{\partial\theta}$, the latter being the tangential component of $\frac{\partial}{\partial \bar{z}}$ and we have the highest weight representation of $U(1)$. In Section 2 we shall introduce the *Cauchy kernel* of the integral representation of zero mode spinors:

$$K^{\dagger}(z, \zeta) = \frac{1}{|\zeta - z|^3} \gamma_-(\zeta - z). \tag{0-2}$$

The integral representation formula of an even spinor is given in Theorem 2.2:

$$\varphi(z) = -\frac{1}{2\pi^2} \int_G K^{\dagger}(z,\zeta) D\varphi(\zeta) dV(\zeta) + \frac{1}{2\pi^2} \int_{\partial G} K^{\dagger}(z,\zeta)(\gamma_{\partial G}\varphi)(\zeta) d\sigma(\zeta), \tag{0-3}$$

where $\gamma_{\partial G}$ is Clifford multiplication of the outer unit normal to ∂G. The higher dimensional Cauchy kernel, as a gradient of the Newtonian kernel, was first obtained by Dixon [Dx] and by Moisil [Moi]. It was made clear in recent works, in particular, [G-M], [C], that it is a chiral operator whenever acting on spinors with chirality. This is transparent by the factor $\gamma_-(\zeta - z)$ in (0-2). Throughout the paper the author emphasized the point of view that the Cauchy kernel defines a transformation with singular kernel between spinors of different chirality, contrary to the description of the Cauchy kernel as Clifford multiplication.

We discuss in Theorem 2.6 the local solution of the equation $D\varphi = \psi$.

In [K-2] we showed that the boundary trace gives the isomorphism of the space of zero mode spinors in the interior (resp. exterior) of unit ball and the space spanned by the eigenspinors corresponding to non-negative (resp. negative) eigenvalues of the tangential Dirac operator on the unit sphere. The inverse is a Bergman-Szegö type integral, that is, eigenspinor expansion by $\phi^{\pm(m,l,k)}$:

$$A^{\pm}\varphi = \frac{1}{2\pi^2}\sum_{m,l,k}\left(\varphi, \phi^{\pm(m,l,k)}\right)\phi^{\pm(m,l,k)}.$$

An analogous theorem was obtained also by Mcintosh [Mc] and Ryan [R-2] in connection with the Hardy space of monogenic functions. In Theorem 3.4 we shall prove that the Bergman-Szegö kernel is essentially the Cauchy kernel.

$$\text{(0-4)}\qquad 2\pi^2 A_{\pm}(z,\zeta) = \pm K^{\dagger}(z,\zeta)\cdot\gamma_+(\zeta), \quad |z| < |\zeta| = 1 \text{ (resp. } |z| > |\zeta| = 1).$$

This is one of our main results. Recall that the Cauchy kernel on C has the expansion:

$$\frac{1}{2\pi}\sum e^{\mp ik\theta} z^{\pm k}\, d\theta = \pm\frac{1}{2\pi i}\frac{d\zeta}{\zeta - z}, \quad |z| < 1 \text{ (resp. } |z| > 1), \ \zeta = e^{i\theta}.$$

The Cauchy integral formula and the expansion (0.4) yields Taylor and Laurent expansions of zero mode spinors (Theorems 3.5 and 4.1). The Taylor and Laurent expansions as well as the expansion of Cauchy kernel were discussed earlier by many Clifford algebraists. Our novel aspect is to give them in a more transparent way by making use of the basis consisting of eigenspinors of the tangential Dirac operator.

In Section 4 we shall introduce *meromorphic spinors*. The *residue* of a meromorphic spinor at $z = c$ is the pair of coefficients that correspond to the terms of the expansion of order $O\left(\frac{1}{|z-c|^3}\right)$. We shall investigate the effect of conformal transformations, and then by Kelvin inversion we shall develop the theory around the point at infinity. Then we prove that the sum of the residues of a meromorphic spinor on S^4 is zero. The change of the basis vectors $\phi^{\pm(m,l,k)}$ under conformal transformations is discussed in 4.3. In Section 5 we shall extend our theory to conformally flat 4-manifolds. We shall prove under some hypothesis that, on a compact conformally flat 4-manifold, the sum of the residues of a meromorphic spinor is zero.

1. Dirac operator and zero mode spinors

1.1. Let $\Delta = \Delta^+ \oplus \Delta^-$ be an irreducible complex representation of the Clifford algebra $\mathrm{Clif}(\mathrm{R}^4)$; $\mathrm{Clif}(\mathrm{R}^4) \otimes \mathrm{C} \equiv \mathrm{End}(\Delta)$. Δ decomposes into irreducible representations $\Delta^{\pm}$ of $\mathrm{Spin}(4)$, each of which has $\dim \Delta^{\pm} = 2$. Let $S = \mathrm{R}^4 \times \Delta$ be the spinor bundle on R^4. The corresponding bundle S^+ (resp. S^-) is called the even (resp. odd) spinor bundle.

We shall choose complex coordinates and make the identification $\mathrm{R}^4 \simeq \mathrm{C}^2$.

The Dirac operator is defined by

$$\mathcal{D} = c \circ d$$

where $d : S \longrightarrow S \otimes T^*\mathrm{C}^2 \simeq S \otimes T\mathrm{C}^2$ is the exterior differentiation and $c : S \otimes T\mathrm{C}^2 \longrightarrow S$ is the bundle homomorphism coming from the Clifford multiplication. By means of the decomposition $S = S^+ \oplus S^-$ the Dirac operator is decomposed into chiral components:

$$\mathcal{D} = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} : C^\infty\left(\mathrm{C}^2, S^+ \oplus S^-\right) \longrightarrow C^\infty\left(\mathrm{C}^2, S^+ \oplus S^-\right). \tag{1-1-1}$$

We find that D and $D^\dagger$ have the following coordinate expressions;

$$D = \begin{pmatrix} \frac{\partial}{\partial z_1} & -\frac{\partial}{\partial \bar{z}_2} \\ \frac{\partial}{\partial z_2} & \frac{\partial}{\partial \bar{z}_1} \end{pmatrix}, \qquad D^\dagger = \begin{pmatrix} \frac{\partial}{\partial \bar{z}_1} & \frac{\partial}{\partial \bar{z}_2} \\ -\frac{\partial}{\partial z_2} & \frac{\partial}{\partial z_1} \end{pmatrix}. \tag{1-1-2}$$

An even (respectively odd) spinor φ is called a *zero mode spinor* if $D\varphi = 0$ (respectively $D^\dagger\varphi = 0$). Several authors call it a monogenic spinor, Clifford regular spinor or harmonic spinor. Monogenic or Clifford regular is used for Clifford algebra valued functions annihilated by the Dirac operator [B-D-S, C, G-M, etc.], while harmonic is used for spinors or functions taking values in a Clifford module [H]. On the other hand, on a compact manifold $D\varphi = 0$ if and only if $\Delta\varphi = D^\dagger D\varphi = 0$. This is not true on a non-compact manifold, so we prefer to call it a zero mode spinor.

1.2. The Lie group $SU(2)$ acts on C^2 from the right and from the left. Let $dR(g)$ and $dL(g)$ denote respectively the right and left infinitesimal actions of the Lie algebra $su(2)$. We define the following vector fields on C^2.

$$\theta_i = -dR\left(\frac{\sqrt{-1}}{2}\sigma^i\right), \qquad \tau_i = dL\left(\frac{\sqrt{-1}}{2}\sigma^i\right), \qquad i = 1, 2, 3, \tag{1-2-1}$$

where σ^1, σ^2 and σ^3 are the Pauli matrices. The vector fields $\theta_i(z)$, $i = 1,2,3$, and $\tau_i(z)$, $i = 1,2,3$ are tangent at z to the sphere $|z| = \text{constant} \neq 0$.

It is more convenient to introduce the vector fields;

$$\begin{aligned} \epsilon &= \theta_1 + \sqrt{-1}\theta_2 = -\overline{z}_2\frac{\partial}{\partial z_1} + \overline{z}_1\frac{\partial}{\partial z_2}, \\ \delta &= \tau_1 + \sqrt{-1}\tau_2 = \overline{z}_2\frac{\partial}{\partial \overline{z}_1} - z_1\frac{\partial}{\partial z_2}. \end{aligned} \tag{1-2-2}$$

The following commutation relation holds on $B = \{|z| = 1\}$.

$$\begin{aligned} &[\theta_3, \epsilon] = \sqrt{-1}\epsilon, \quad [\theta_3, \overline{\epsilon}] = -\sqrt{-1}\overline{\epsilon}, \quad [\epsilon, \overline{\epsilon}] = 2\sqrt{-1}\theta_3, \\ &[\tau_3, \delta] = \sqrt{-1}\delta, \quad [\tau_3, \overline{\delta}] = -\sqrt{-1}\overline{\delta}, \quad [\delta, \overline{\delta}] = 2\sqrt{-1}\tau_3. \end{aligned} \tag{1-2-3}$$

In the following we denote a function $f(z,\overline{z})$ of variables z, $\overline{z}$ simply by $f(z)$. We put, for $m = 0,1,2,\dots$ and $l,k = 0,1,\dots,m$,

$$\begin{aligned} h_{(m,l,k)}(z) &= \epsilon^k(z_1^l z_2^{m-l}), \\ \widehat{h}_{(m,l,k)}(z) &= \delta^k(\overline{z}_1^l z_2^{m-l}). \end{aligned} \tag{1-2-4}$$

Then $h_{(m,l,k)}$ and $\widehat{h}_{(m,l,k)}$ are harmonic polynomials on $\mathbb{C}^2$; $\Delta h_{(m,l,k)} = \left(\frac{\partial^2}{\partial z_1 \partial \overline{z}_1} + \frac{\partial^2}{\partial z_2 \partial \overline{z}_2}\right) h_{(m,l,k)} = 0$ etc., here Δ is the Laplacian. Moreover they form a linear basis of the space of harmonic polynomials.

We have the following relations:

$$\begin{aligned} \sqrt{-1}\theta_3\, h_{(m,l,k)} &= \left(\frac{m}{2} - k\right) h_{(m,l,k)}, \\ \epsilon\, h_{(m,l,k)} &= h_{(m,l,k+1)}, \\ \overline{\epsilon}\, h_{(m,l,k)} &= -k(m-k+1) h_{(m,l,k-1)}, \\ \sqrt{-1}\tau_3\, \widehat{h}_{(m,l,k)} &= \left(\frac{m}{2} - k\right) \widehat{h}_{(m,l,k)}, \\ \delta\, \widehat{h}_{(m,l,k)} &= \widehat{h}_{(m,l,k+1)}, \\ \overline{\delta}\, \widehat{h}_{(m,l,k)} &= -k(m-k+1) \widehat{h}_{(m,l,k-1)}. \end{aligned} \tag{1-2-5}$$

Therefore the space of harmonic polynomials on $\mathbb{C}^2$ is decomposed by the right action R of $SU(2)$ into $\sum_m \sum_{l=0}^m H_{m,l}$, each $H_{m,l} = \sum_{k=0}^m \mathbb{C} h_{(m,l,k)}$ being an $(m+1)$ dimensional irreducible representation with highest weight $\frac{m}{2}$. Similarly the space of harmonic polynomials on $\mathbb{C}^2$ is decomposed by the left action L of $SU(2)$ into $\sum_m \sum_{l=0}^m \widehat{H}_{m,l}$, each $\widehat{H}_{m,l} = \sum_{k=0}^m \mathbb{C}\widehat{h}_{(m,l,k)}$ being an $(m+1)$ dimensional irreducible representation with highest weight $\frac{m}{2}$, [K1, G-H].

1.3. We shall introduce a set of zero mode even spinors which, restricted to $B=\{|z|=1\}$, forms a complete orthonormal basis of $L^2(B,S^+)$. These zero mode spinors and their variants are important in our theory just as the monomial z^n or its inverse $\frac{1}{z^n}$ are important in complex function theory.

We put, for $m=0,1,2,\dots$, $l=0,1,\dots,m$ and $k=0,1,\dots,m+1$,

$$\begin{aligned}\phi^{(m,l,k)}(z)&=\sqrt{\frac{(m+1-k)!}{k!l!(m-l)!}}\begin{pmatrix}kh_{(m,l,k-1)}\\ -h_{(m,l,k)}\end{pmatrix},\\ \phi^{-(m,l,k)}(z)&=\sqrt{\frac{(m+1-k)!}{k!l!(m-l)!}}\left(\frac{1}{|z|^2}\right)^{m+2}\begin{pmatrix}\widehat{h}_{(m+1,l,k)}\\ \widehat{h}_{(m+1,l+1,k)}\end{pmatrix}.\end{aligned}\tag{1-3-1}$$

Then $\phi^{(m,l,k)}\in C^\infty(\mathbf{C}^2,S^+)$ and $\phi^{-(m,l,k)}\in C^\infty(\mathbf{C}^2\setminus\{0\},S^+)$, and we have the following homogeneity relations;

$$\begin{aligned}\phi^{(m,l,k)}(z)&=|z|^m\phi^{(m,l,k)}\left(\frac{z}{|z|}\right)\\ \phi^{-(m,l,k)}(z)&=|z|^{-(m+3)}\phi^{-(m,l,k)}\left(\frac{z}{|z|}\right),\end{aligned}\tag{1-3-2}$$

for $z\neq 0$.

For any $c\in\mathbf{C}^2$ we put

$$\phi_c^{\pm(m,l,k)}(z)=\phi^{\pm(m,l,k)}(z-c).\tag{1-3-3}$$

The following proposition is proved by a direct calculation.

PROPOSITION 1.1.

$$\begin{aligned}D\phi_c^{(m,l,k)}(z)&=0,\quad on\ \ \mathbf{C}^2.\\ D\phi_c^{-(m,l,k)}(z)&=0,\quad on\ \ \mathbf{C}^2\setminus\{c\}.\end{aligned}\tag{1-3-4}$$

1.4. Let ν and μ be vector fields on $\mathbf{C}^2$ defined by

$$\nu=z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2},\qquad \mu=z_2\frac{\partial}{\partial z_2}+\overline{z}_1\frac{\partial}{\partial\overline{z}_1}.\tag{1-4-1}$$

Then the radial vector field is defined by

$$\frac{\partial}{\partial n}=\frac{1}{2|z|}(\nu+\overline{\nu})=\frac{1}{2|z|}(\mu+\overline{\mu}),\qquad |z|\neq 0.\tag{1-4-2}$$

We have also $\theta_3 = \frac{1}{2\sqrt{-1}}(\nu - \overline{\nu})$, $\tau_3 = \frac{1}{2\sqrt{-1}}(\mu - \overline{\mu})$. Each quartet $\{\frac{\partial}{\partial n}, \frac{1}{|z|}\theta_i,\ i = 1,2,3\}$ and $\{\frac{\partial}{\partial n}, \frac{1}{|z|}\tau_i,\ i = 1,2,3\}$ forms an orthogonal frame on $\mathbf{C}^2 \setminus \{0\}$. The Dirac operator written in this orthogonal frame is

$$\mathcal{D} = \left(\frac{\partial}{\partial n}\right) \cdot \nabla_{(\frac{\partial}{\partial n})} + \sum_{i=1}^{3} \left(\frac{1}{|z|}\theta_i\right) \cdot \nabla_{\frac{1}{|z|}\theta_i} \quad |z| \neq 0. \tag{1-4-3}$$

The second term gives the (chiral) tangential Dirac operator on the sphere $B_r = \{|z| = r\}$.

On a smooth boundary ∂G of a region G we shall denote by $\gamma_{\partial G}$ the Clifford multiplication of the outer unit normal to ∂G. We shall abbreviate it as γ if it is obvious which boundary we are considering. In particular, γ_0 denotes the Clifford multiplication of the radial vector $\frac{\partial}{\partial n}$. The Clifford multiplication of $\gamma_{\partial G}$ changes the chiralty:

$$\gamma_{\partial G} : S^+ \oplus S^- \longrightarrow S^- \oplus S^+; \quad \gamma_{\partial G}^2 = 1. \tag{1-4-4}$$

In particular the matrix expression of γ_0 becomes as follows.

$$\gamma_0|S^+ = \frac{1}{|z|}\begin{pmatrix} \overline{z}_1 & -z_2 \\ \overline{z}_2 & z_1 \end{pmatrix}, \qquad \gamma_0|S^- = \frac{1}{|z|}\begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix}. \tag{1-4-5}$$

In the sequel we shall write γ_+ (resp. γ_-) for $\gamma_0|S^+$ (resp. $\gamma_0|S^-$).

PROPOSITION 1.2. *The Dirac operators D and $D^\dagger$ have the following polar decompositions*:

$$\begin{aligned} D &= \gamma_+\left(\frac{\partial}{\partial n} - \partial\!\!\!/\right), \\ D^\dagger &= \left(\frac{\partial}{\partial n} + \partial\!\!\!/ + \frac{3}{2|z|}\right)\gamma_-, \end{aligned} \tag{1-4-6}$$

where the tangential (nonchiral) Dirac operator $\partial\!\!\!/$ is given by

$$\partial\!\!\!/ = -(\gamma_-)\left[\sum_{i=1}^{3}\left(\frac{1}{|z|}\theta_i\right) \cdot \nabla_{\frac{1}{|z|}\theta_i}\right] = \frac{1}{|z|}\begin{pmatrix} -\sqrt{-1}\theta_3 & \overline{\epsilon} \\ -\epsilon & \sqrt{-1}\theta_3 \end{pmatrix}.$$

PROOF. We have $\frac{\partial}{\partial z_1} = \frac{1}{|z|^2}(\overline{z}_1\nu - z_2\epsilon)$, etc. Other components of D and $D^\dagger$ in the expression (1-1-2) are calculated similarly and we have the desired formulas.

Let $B = \{|z| = 1\}$. The tangential Dirac operator

$$\partial\!\!\!/ \,|\, B : C^\infty(B, S^+) \longrightarrow C^\infty(B, S^+) \tag{1-4-7}$$

is a self adjoint elliptic differential operator. By virtue of the properties (1-2-5) of the polynomials $h_{(m,l,k)}$ and $\widehat{h}_{(m,l,k)}$ we can solve the eigenvalue problem for $\partial\!\!\!/$, [K-2, Sch].

PROPOSITION 1.3. *On $B = \{|z| = 1\}$ we have;*
(1)

$$\partial\!\!\!/\phi^{(m,l,k)} = \frac{m}{2}\phi^{(m,l,k)},$$
$$\partial\!\!\!/\phi^{-(m,l,k)}(z) = -\frac{m+3}{2}\phi^{-(m,l,k)},$$

(2) *the eigenvalues of $\partial\!\!\!/$ are*

$$\frac{n}{2}, \quad -\frac{n+3}{2}; \quad n = 0, 1, \ldots,$$

and the multiplicity of each eigenvalue is equal to $(n+1)(n+2)$,
(3) *the set of eigenspinors*

$$\left\{\frac{1}{\sqrt{2}\pi}\phi^{(m,l,k)}, \quad \frac{1}{\sqrt{2}\pi}\phi^{-(m,l,k)}; \quad m = 0, 1, \ldots, \quad k, l = 0, 1, \ldots, m+1\right\}$$

forms a complete orthonormal system of $L^2(B, S^+)$.

The completeness follows from the fact that the spinors $\begin{pmatrix} h_{(k,l,m)} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ h_{(k,l,m)} \end{pmatrix}$ are in linear span of $\phi^{\pm(k,l,m)}$'s and the fact that the harmonic polynomials $h_{(k,l,m)}$ restricted on B are complete in the space of square-integrable functions on B. The constant for normalization in (1-3-1) is determined by the integral:

$$\int_B |z_1^a z_2^b|^2 d\sigma_3 = 2\pi^2 \frac{a!b!}{(a+b+1)!},$$

where σ_3 is the surface measure of the unit sphere $B = \{|z| = 1\}$;

$$\int_B d\sigma_3 = 2\pi^2.$$

1.5. For odd spinors and the Dirac operator $D^\dagger$ the parallel argument as above is valid.

We put

$$\begin{aligned} \psi^{(m,l,k)}(z) &= |z|^{-(2m+3)}\gamma_+(z)\phi^{(m,l,k)}(z), \quad \text{for } z \in \mathbf{C}^2 \setminus 0, \\ \psi^{-(m,l,k)}(z) &= |z|^{(2m+3)}\gamma_+(z)\phi^{-(m,l,k)}(z), \quad \text{for } z \in \mathbf{C}^2. \end{aligned} \tag{1-5-1}$$

We have

$$\psi^{-(m,l,k)}(z) \in C^\infty(\mathbf{C}^2, S^-), \quad \psi^{(m,l,k)}(z) \in C^\infty(\mathbf{C}^2 \setminus \{0\}, S^-).$$

The following homogeneity relations hold.

(1-5-2)
$$\psi^{-(m,l,k)}(z) = |z|^m \psi^{-(m,l,k)}\left(\frac{z}{|z|}\right), \quad \psi^{(m,l,k)}(z) = |z|^{-(m+3)} \psi^{(m,l,k)}\left(\frac{z}{|z|}\right).$$

The translation by c is denoted as

(1-5-3) $$\psi_c^{(m,l,k)}(z) = \psi^{(m,l,k)}(z-c), \quad \psi_c^{-(m,l,k)}(z) = \psi^{-(m,l,k)}(z-c).$$

We have

PROPOSITION 1.4.

(1-5-4)
$$\begin{aligned} D^\dagger \psi_c^{-(m,l,k)}(z) = 0 \quad &on \;\; \mathbf{C}^2. \\ D^\dagger \psi_c^{(m,l,k)}(z) = 0 \quad &on \;\; \mathbf{C}^2 \setminus \{c\}. \end{aligned}$$

The proof follows by a direct calculation.

REMARK. The eigenspace of the tangential Dirac operator acting either on Clifford algebra valued functions or on the spinors on the sphere $S^m \subset \mathbf{R}^{m+1}$ was investigated in the literatures [B-D-S, D-S-Sč, S, So-1]. They called the eigenvectors corresponding to the eigenvalue $\pm\frac{k}{2}$ *the inner* (*resp. outer*) *spherical monogenics of order* $\pm k$. The orthonormal basis of inner spherical monogenics of order k are given by

(1-5-5) $$V_{i_1\cdots i_k}(x) = \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k}^{m} (x_{i_1}e_0 - x_0 e_{i_1})(x_{i_2}e_0 - x_0 e_{i_2}) \cdots (x_{i_k}e_0 - x_0 e_{i_k}),$$

while the orthonormal basis of outer spherical monogenics of order $-(k+3)$ are given by

(1-5-6) $$W_{i_1\cdots i_k}(x) = (-1)^k \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}} \left(\frac{\overline{x}}{|x|^{m+1}}\right), \quad x = \sum_{i=0}^{m} x_i e_i.$$

The reader can rewrite these formulas in case of $m = 3$ into the almost same expressions as our $\phi^{\pm(m,l,k)}$ written in polynomial forms.

1.6. The inner product of two spinors ϕ and ψ is defined by $\langle \phi\,, \psi \rangle = \sum_{k=1}^{4} \phi_k \overline{\psi}_k$. A vector $\gamma \in \mathrm{C}^2$ of length one acts as a unitary transformation on spinors:

$$\langle \gamma\phi\,, \gamma\psi \rangle = \langle \phi\,, \psi \rangle. \tag{1-6-1}$$

Let G be a domain in C^2 with smooth boundary ∂G. The surface measure on ∂G is denoted by $d\sigma_G$, or, in brief, $d\sigma$. For $\phi \in C^\infty(\overline{G}, S^+)$ and $\psi \in C^\infty(\overline{G}, S^-)$, we have the following Stokes' formula;

$$\int_G \langle D\phi, \psi \rangle \, dV + \int_G \langle \phi, D^\dagger \psi \rangle \, dV = \int_{\partial G} \langle \gamma\, \phi, \psi \rangle \, d\sigma, \tag{1-6-2}$$

where $\gamma = \gamma_{\partial G} | S^+$ is the Clifford multiplication of the outer unit normal to ∂G.

THEOREM 1.5 (Cauchy's integral theorem). *Let $\varphi \in C^\infty(\overline{G}, S^+)$ be a zero mode spinor for D. Then we have*

$$\int_{\partial G} \langle \gamma\, \varphi,\, \psi^{-(m,l,k)} \rangle \, d\sigma = 0,$$

for any (m, l, k). In particular

$$\int_{\partial G} \gamma\, \varphi d\sigma = 0. \tag{1-6-3}$$

PROOF. The first assertion is a direct consequence of Stokes' formula and Proposition 1.4. The second assertion follows if we take, in particular,

$$\psi^{-(0,0,1)} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \psi^{-(0,0,0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

LEMMA 1.6. *Let $\varphi \in C^\infty(\overline{G}, S^+)$. If $D\varphi = 0$ and if the boundary trace $\varphi | \partial G = 0$ then $\varphi \equiv 0$.*

In fact if $D\varphi = 0$ then $|\varphi| = \sqrt{\langle \varphi, \varphi \rangle}$ is a subharmonic function on G hence $|\varphi|$ takes its maximum on the boundary. We know that the Dirac operator has the unique continuation property, [B-W].

PROPOSITION 1.7. *Let $\varphi \in C^\infty(\overline{G}, S^+)$. If $D\varphi = 0$ and $D^\dagger(\gamma\varphi) = 0$, then $\varphi \equiv 0$.*

PROOF. From Stokes' formula we have

$$\begin{aligned} 0 &= \int_G \langle D\varphi, \gamma\varphi \rangle \, dV + \int_G \langle \varphi, D^\dagger \gamma\varphi \rangle \, dV = \int_{\partial G} \langle \gamma\, \varphi, \gamma\, \varphi \rangle \, d\sigma \\ &= \int_{\partial G} \langle \varphi, \varphi \rangle \, d\sigma. \end{aligned}$$

Hence $\varphi = 0$ on the boundary. From the lemma it follows that $\varphi \equiv 0$.

1.7. The following theorem is proved by Y. Homma [Ho].

THEOREM 1.8. *$L^2(B, S^+)$ is decomposed by the two-sided action of $SU(2) \times SU(2) = \mathrm{Spin}(4)$ into $\sum_{m=-\infty}^{\infty} H_m$, where $H_{\pm m} = \sum_{l=0}^{m} \sum_{k=0}^{m+1} \mathbb{C}\phi^{\pm(m,l,k)}$ is an $(m+1)(m+2)$ dimensional irreducible representation with highest weight $\frac{m+1}{2}$.*

The left action of σ^i, $i = 1, 2, 3$, on an even spinor is given by the derivation $dL(\frac{\sqrt{-1}}{2}\sigma^i) = \tau_i$, while the right action is given by $dR(\frac{\sqrt{-1}}{2}\sigma^i) + \frac{\sqrt{-1}}{2}\sigma^i = -\theta_i + \frac{\sqrt{-1}}{2}\sigma^i$. These actions commute with $\partial\!\!\!/$. Put

$$\sqrt{-1}T = \sqrt{-1}\theta_3 - \frac{1}{2}\sigma^3, \quad E = \epsilon - \frac{\sqrt{-1}}{2}(\sigma^2 - \sqrt{-1}\sigma^1), \quad F = \bar{\epsilon} - \frac{\sqrt{-1}}{2}(\sigma^2 + \sqrt{-1}\sigma^1).$$

Then we have the following relations:

$$\begin{aligned}
&\sqrt{-1}T\phi^{(m,l,k)} = \Big(\frac{m+1}{2} - k\Big)\phi^{(m,l,k)}, && \sqrt{-1}\tau_3\phi^{(m,l,k)} = \Big(m - \frac{l}{2}\Big)\phi^{(m,l,k)},\\
&E\phi^{(m,l,k)} = \phi^{(m,l,k+1)}, && \delta\phi^{(m,l,k)} = -(m-l)\phi^{(m,l+1,k)},\\
&F\phi^{(m,l,k)} = -k(m-k+2)\phi^{(m,l,k-1)}, && \bar{\delta}\phi^{(m,l,k)} = l\phi^{(m,l-1,k)},\\
&\sqrt{-1}T\phi^{-(m,l,k)} = \Big(m - \frac{l}{2}\Big)\phi^{-(m,l,k)}, && \sqrt{-1}\tau_3\phi^{-(m,l,k)} = \Big(\frac{m+1}{2} - k\Big)\phi^{-(m,l,k)},\\
&E\phi^{-(m,l,k)} = (m-l)\phi^{-(m,l+1,k)}, && \delta\phi^{-(m,l,k)} = \phi^{-(m,l,k+1)},\\
&F\phi^{-(m,l,k)} = -l\phi^{-(m,l-1,k)}, && \bar{\delta}\phi^{-(m,l,k)} = -k(m-k+2)\phi^{-(m,l,k-1)},
\end{aligned}$$

$$k = 0, 1, \ldots, m+1, \; l = 0, 1, \ldots, m.$$

This gives a representation of Spin(4) and the highest weight vector is $\phi^{(m,m,m+1)}$.

2. Cauchy's integral formula

2.1. The Cauchy kernel was defined as Clifford multiplication of the radial derivative of the Newton kernel in Section 11.4 of [B-D-S]. So we put, for any pair (z, ζ) such that $\zeta \neq z$,

$$\mathcal{K} = \frac{1}{|\zeta - z|^3}\gamma_0(\zeta - z) : C^\infty(\mathbb{C}^2, S) \longrightarrow C^\infty(\mathbb{C}^2, S). \tag{2-1-1}$$

$\mathcal{K}$ has the following matrix representation.

$$\mathcal{K} = \begin{pmatrix} 0 & K^\dagger \\ K & 0 \end{pmatrix}$$

$$\text{(2-1-2)} \qquad K^\dagger(z,\zeta) = \frac{1}{|\zeta - z|^3}\gamma_-(\zeta - z) = \frac{1}{|\zeta - z|^4}\begin{pmatrix} \zeta_1 - z_1 & \zeta_2 - z_2 \\ -\overline{\zeta}_2 + \overline{z}_2 & \overline{\zeta}_1 - \overline{z}_1 \end{pmatrix},$$
$$K(z,\zeta) = \frac{1}{|\zeta - z|^3}\gamma_+(\zeta - z).$$

PROPOSITION 2.1.

$$\text{(2-1-3)} \qquad D_z K^\dagger(z,\zeta) = 0, \qquad D_z^\dagger K(z,\zeta) = 0.$$

As was mentioned in the introduction the Cauchy's integral formula for a zero mode spinor has been obtained by many authors both in Clifford analysis and in spinor analysis. Our approach is to specialize it to the irreducible spinor bundles. It has an advantage to make clear the spaces where the Cauchy transformation acts, and it is a necessary procedure for treating the Dirac equation as you will see in the next paragraph.

THEOREM 2.2 (Integral formula). *Let G be a domain in $\mathbf{C}^2$ and let $\varphi \in C^\infty(\overline{G}, S^+)$. Then we have*

$$\text{(2-1-4)} \qquad \begin{aligned} \varphi(z) = &-\frac{1}{2\pi^2}\int_G K^\dagger(z,\zeta)D\varphi(\zeta)dV(\zeta) \\ &+ \frac{1}{2\pi^2}\int_{\partial G} K^\dagger(z,\zeta)(\gamma\varphi)(\zeta)d\sigma(\zeta), \qquad z \in G, \end{aligned}$$

where $\gamma = \gamma_{\partial G}|S^+$ is Clifford multiplication of the outer unit normal to ∂G and σ is the surface measure on ∂G.

PROOF. From the Stokes' formula and Proposition 2.1 we have

$$\begin{aligned} \int_{G\cap\{|\zeta - z|\geq\epsilon\}} K^\dagger(z,\zeta)D\varphi(\zeta)dV(\zeta) &= \int_{\partial G} K^\dagger(z,\zeta)(\gamma\varphi)(\zeta)d\sigma(\zeta) \\ &\quad - \int_{|\zeta - z| = \epsilon} K^\dagger(z,\zeta)\gamma(\zeta)\varphi(\zeta)d\sigma(\zeta). \end{aligned}$$

On $|\zeta - z| = \epsilon$ we have

$$K^\dagger(z,\zeta)\gamma(\zeta) = \frac{1}{\epsilon^3}\gamma_-(\zeta - z)\gamma_{B_\epsilon}(\zeta) = \frac{1}{\epsilon^3}Id_{S^+}.$$

The formula follows from the fact;

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^3}\int_{|\zeta-z|=\epsilon}\varphi(\zeta)d\sigma(\zeta)=2\pi^2\varphi(z).$$

PROPOSITION 2.3 (Cauchy's integral formula). *Let G be a domain in $\mathbf{C}^2$ with a smooth boundary ∂G and let $\varphi\in C^\infty(\overline{G},S^+)$ be such that $D\varphi=0$. Then we have*

$$\varphi(z)=\frac{1}{2\pi^2}\int_{\partial G}K^\dagger(z,\zeta)(\gamma\varphi)(\zeta)d\sigma(\zeta),\quad z\in G. \tag{2-1-5}$$

We see from Corollary 3.2 of the next section that an even zero mode spinor φ that is square integrable away from a compact set satisfies the decay condition $|\varphi|\sim O(\frac{1}{|z|^3})$ as $|z|\longrightarrow\infty$. Hence we have the following integral formula on an exterior domain.

PROPOSITION 2.4. *Let $G=\mathbf{C}^2\backslash K$ be an exterior domain with smooth boundary and let $\varphi\in C^\infty(\overline{G},S^+)\cap L^2(\overline{G},S^+)$ be such that $D\varphi=0$. Then we have*

$$\varphi(z)=-\frac{1}{2\pi^2}\int_{\partial G}K^\dagger(z,\zeta)(\gamma\varphi)(\zeta)d\sigma(\zeta),\quad z\in G. \tag{2-1-6}$$

2.2. Local solution of $D\varphi=\psi$

We denote the inner product of two spinors by

$$(\varphi_1,\varphi_2)=\int_{\mathbf{C}^2}\langle\varphi_1,\varphi_2\rangle\,dV\quad\text{for}\quad\varphi_1,\varphi_2\in C^\infty(\mathbf{C}^2,S).$$

For $\varphi\in C_c^\infty(\mathbf{C}^2,S^\pm)$ we put

$$K\varphi(z)=\int_{\mathbf{C}^2}K(z,\zeta)\varphi(\zeta)dV(\zeta)$$
$$K^\dagger\varphi(z)=\int_{\mathbf{C}^2}K^\dagger(z,\zeta)\varphi(\zeta)dV(\zeta).$$

These are well defined out of $\operatorname{Supp}\varphi$, and are of zero mode.

LEMMA 2.5.

$$(\varphi,K^\dagger\psi)=(K\varphi,\psi), \tag{2-2-1}$$

for $\varphi\in C_c^\infty(\mathbf{C}^2,S^+)$ and $\psi\in C_c^\infty(\mathbf{C}^2,S^-)$.

In fact

$$\langle \varphi(z), \gamma_-(\zeta - z)\psi(\zeta)\rangle = \langle \gamma_+(\zeta - z)\varphi(z), \psi(\zeta)\rangle$$

is bounded by a constant that depends only on φ and ψ. The lemma follows from the formula (2-1-2).

THEOREM 2.6. *Given a $\psi \in C_c^\infty(\mathrm{C}^2, S^-)$, there is a solution $\phi \in C^\infty(\mathrm{C}^2, S^+)$ of the equation*

$$D\phi(z) = \psi(z), \qquad z \in \mathrm{C}^2. \tag{2-2-2}$$

PROOF. First we note the following integral formula for an odd spinor $\varphi \in C^\infty(\mathrm{C}^2, S^-)$, that is similar to (2-1-4):

(2-2-3)

$$\varphi(z) = -\frac{1}{2\pi^2}\int_G K(z,\zeta)D^\dagger\varphi(\zeta)dV(\zeta) + \frac{1}{2\pi^2}\int_{\partial G} K(z,\zeta)(\gamma\varphi)(\zeta)d\sigma(\zeta), \quad z \in G,$$

where γ means $\gamma_{\partial G}|S^-$. In particular

$$\varphi(z) = -\frac{1}{2\pi^2}KD^\dagger\varphi(z) = -\frac{1}{2\pi^2}\int_{\mathrm{C}^2} K(z,\zeta)D^\dagger\varphi(\zeta)dV(\zeta), \tag{2-2-4}$$

for a $\varphi \in C_c^\infty(\mathrm{C}^2, S^-)$. Now let $\psi \in C_c^\infty(\mathrm{C}^2, S^-)$ and put $\phi = \frac{1}{2\pi^2}K^\dagger\psi$. We have $\phi \in C^\infty(\mathrm{C}^2, S^+)$. From Stokes' formula we have, for any $\varphi \in C_c^\infty(\mathrm{C}^2, S^-)$,

$$(D\phi, \varphi) = -(\phi, D^\dagger\varphi) = -\frac{1}{2\pi^2}(K^\dagger\psi, D^\dagger\varphi),$$

which is equal to $-\frac{1}{2\pi^2}(\psi, KD^\dagger\varphi)$ from the above lemma. By virtue of (2-2-4) we have

$$(D\phi, \varphi) = -\frac{1}{2\pi^2}(\psi, KD^\dagger\varphi) = (\psi, \varphi).$$

The assertion is proved.

3. Expansion of the Cauchy kernel

3.1. Bergman-Szegö type integral

We shall deal with the following problem.

Given an even spinor on a sphere, can we extend it to the interior (resp. exterior) of the ball as a zero mode spinor?

This problem was discussed in [K2]. Here we shall present it in a form modified to suit our situation.

We put

$$E_r(c) = \{z \in \mathrm{C}^2;\ |z - c| \le r\}, \qquad B_r(c) = \{z \in \mathrm{C}^2;\ |z - c| = r\}.$$

The results we have developed in Section 1 on the unit ball and on its boundary are also valid on $E_r(c)$ and on $B_r(c)$ if we translate the quantities according to the variable change $z \longrightarrow \frac{z-c}{r}$. The surface measure of $B_r(c)$ is given by $d\sigma = r^3 d\sigma_3$.

The tangential Dirac operator on $B_r(c)$ becomes

$$\partial\!\!\!/_{(c,r)} = \frac{r}{|z-c|}\begin{pmatrix} -\sqrt{-1}\theta_3(z-c) & \bar{\epsilon}(z-c) \\ -\epsilon(z-c) & \sqrt{-1}\theta(z-c) \end{pmatrix}, \tag{3-1-1}$$

where $\epsilon(z-c) = -(\overline{z_2 - c_2})\frac{\partial}{\partial z_1} + (\overline{z_1 - c_1})\frac{\partial}{\partial z_2}$, etc.

The eigenvalues of $\partial\!\!\!/_{(c,r)}$ are $\frac{n}{2}$, $-\frac{n+3}{2}$, $n = 0, 1, \ldots$. The eigenspinors $\phi_c^{\pm(m,l,k)}$ form a complete orthogonal system in $L^2(B_r(c), S^+)$. The normalized basis vectors are

$$\frac{1}{\sqrt{2}\pi}\phi_{(c,r)}^{\pm(m,l,k)} = \frac{1}{\sqrt{2}\pi} r^{\mp(m+\frac{3}{2})}\phi_c^{\pm(m,l,k)}. \tag{3-1-2}$$

Let $H_\pm$ be the closed subspaces of $L^2(B_r(c), S^+)$ spanned by the eigenvectors $\phi_{(c,r)}^{(m,l,k)}$ (resp. $\phi_{(c,r)}^{-(m,l,k)}$). Let $H^s(E_r(c), S^+)$ denote the Sobolev space of even spinors on $E_r(c)$ with derivatives up to order s in L^2, and similarly for $H^s(B_r(c), S^+)$. Let $b_\pm$ be the trace on the boundary from the interior and from the exterior respectively:

$$\begin{aligned} b_+ &: H^1(E_r(c), S^+) \longrightarrow H^{\frac{1}{2}}(B_r(c), S^+), \\ b_- &: H^1(\mathbf{C}^2 \setminus E_r(c), S^+) \longrightarrow H^{\frac{1}{2}}(B_r(c), S^+). \end{aligned} \tag{3-1-3}$$

We see from (1-3-2) that $\phi_{(c,r)}^{(m,l,k)}$ is in $H^1(E_r(c), S^+)$, while $\phi_{(c,r)}^{-(m,l,k)}$ lies in $H^1(\mathbf{C}^2 \setminus E_r(c), S^+)$.

The Bergman-Szegö type integrals which give the inverse of b are defined as follows. For a $\varphi \in H_+$ we define

$$A^+\varphi(z) = \frac{1}{2\pi^2}\sum_{m,l,k}\left(\varphi\,,\,\phi_{(c,r)}^{(m,l,k)}\right)\phi_{(c,r)}^{(m,l,k)}(z), \quad |z-c| \le r. \tag{3-1-4}$$

Similarly, for a $\varphi \in H_-$ we define

$$A^-\varphi(z) = \frac{1}{2\pi^2}\sum_{m,l,k}\left(\varphi\,,\,\phi_{(c,r)}^{-(m,l,k)}\right)\phi_{(c,r)}^{-(m,l,k)}(z), \quad |z-c| \ge r. \tag{3-1-5}$$

Here the convergence is taken in the sense of norm in each Sobolev space, but, since $|\phi_{(c,r)}^{\pm(m,l,k)}|^2$ are subharmonic functions on $\mathbf{C}^2 \setminus \{c\}$ and since

$$\frac{\partial}{\partial n}|\phi_{(c,r)}^{(m,l,k)}|^2 > 0, \qquad \frac{\partial}{\partial n}|\phi_{(c,r)}^{-(m,l,k)}|^2 < 0,$$

each $|\phi^{\pm(m,l,k)}_{(c,r)}|^2$ takes its maximum on $B_r(c)$, the series $A^+\varphi$ (resp. $A^-\varphi$) converges uniformly for $|z-c|\leq r$ (resp. for $|z-c|\geq r$).

Let

$$\mathcal{N}(E_r(c)) = \{\varphi \in H^1(E_r(c), S^+);\ D\varphi = 0\}$$
$$\mathcal{N}(\mathrm{C}^2 \setminus E_r(c)) = \{\psi \in H^1(\mathrm{C}^2 \setminus E_r(c), S^+);\ D\psi = 0\}.$$

THEOREM 3.1. *The boundary trace gives the following isomorphisms;*

$$\text{(3-1-6)} \qquad \begin{aligned} b_+ &: \mathcal{N}(E_r(c)) \longrightarrow H_+ \cap H^{\frac{1}{2}}(B_r(c), S^+), \\ b_- &: \mathcal{N}(\mathrm{C}^2 \setminus E_r(c)) \longrightarrow H_- \cap H^{\frac{1}{2}}(B_r(c), S^+). \end{aligned}$$

The inverses are given by $A^{\pm}$ respectively.

For the proof see Theorem 5.10 of [K-2].

The similar formulae for the space of square integrable monogenic functions over Liapunov surfaces was obtained in [G-Sp] and that over subdomains of spheres and hyperbolae in [L-M-Q, R].

From (1-3-2) we have the following estimate at infinity.

COROLLARY 3.2. *Let $\varphi \in \mathcal{N}(\mathrm{C}^2 \setminus E_r(c))$. Then*

$$\text{(3-1-7)} \qquad |\varphi(z)| \sim O\left(\frac{1}{|z|^3}\right), \qquad |z| \longrightarrow \infty.$$

We shall now rewrite the expansion of $A^{\pm}\varphi$ in the series of $\phi_c^{\pm(m,l,k)}$ and put the dependence on r into the coefficients, which is a necessary procedure for obtaining the Laurent expansion in the next section, where we will show that the coefficients are also independent of r.

LEMMA 3.3. *Let $\psi^{\pm(m,l,k)}$ be the odd spinors given in* (1-5-1).

(1) *For $\varphi \in H_+$, we have*

$$A^+\varphi(z) = \sum_{m,l,k} C^{(m,l,k)}_{(c,r)}\, \phi^{(m,l,k)}(z-c), \quad |z-c| \leq r,$$

with

$$\text{(3-1-8)} \qquad C^{(m,l,k)}_{(c,r)} = \frac{1}{2\pi^2}\int_{B_r(c)} \langle \gamma_+(\zeta - c)\varphi(\zeta)\,,\, \psi^{(m,l,k)}(\zeta - c)\rangle\, \sigma(d\zeta).$$

(2) *For $\varphi \in H_-$, we have*

$$A^-\varphi(z) = \sum_{m,l,k} C^{-(m,l,k)}_{(c,r)}\, \phi^{-(m,l,k)}(z-c), \quad |z-c| \geq r,$$

with

$$C^{-(m,l,k)}_{(c,r)} = \frac{1}{2\pi^2}\int_{B_r(c)} \langle \gamma_+(\zeta - c)\varphi(\zeta)\,,\, \psi^{-(m,l,k)}(\zeta - c)\rangle\, \sigma(d\zeta). \tag{3-1-9}$$

Proof. Let $\varphi \in H_\pm$. From (3-1-4) and (3-1-5) we have respectively:

$$A^\pm\varphi(z) = \frac{1}{2\pi^2}\sum_{m,l,k} (\varphi(\zeta)\,,\, |\zeta - c|^{\mp(2m+3)}\phi^{\pm(m,l,k)}(\zeta - c))\, \phi^{\pm(m,l,k)}(z - c).$$

By (1-5-1) we can rewrite the coefficients as follows:

$$\begin{aligned}&\frac{1}{2\pi^2}(\varphi(\zeta)\,,\, |\zeta - c|^{\mp(2m+3)}\phi^{\pm(m,l,k)}(\zeta - c))\\ &\quad = \frac{1}{2\pi^2}(\gamma_+(\zeta - c)\varphi(\zeta)\,,\psi^{\pm(m,l,k)}(\zeta - c)) = C^{\pm(m,l,k)}_{(c,r)}.\end{aligned}$$

The lemma is proved.

3.2. The Bergman-Szegö kernel and the Cauchy kernel

Let $A^+_c(z,\zeta)$ be the kernel function of A^+;

$$A^+\phi(z) = \int_{B_r(c)} A^+_c(z,\zeta)\phi(\zeta)d\sigma(\zeta), \qquad z \in E_r(c). \tag{3-2-1}$$

From Subsections 2.1 and 3.1 we know that both $A^+_c(z,\zeta)$ and $K^\dagger(z,\zeta)\cdot\gamma_+(\zeta)$ give the extension of spinors in H_+ to zero mode spinors on $|z-c| < r$. In fact we shall find that the two kernels coincide.

Theorem 3.4 (1). *For $|\zeta - c| = r$ and $|z - c| < r$ it holds that*

$$2\pi^2\, A^+_c(z,\zeta) = K^\dagger(z,\zeta)\cdot\gamma_+(\zeta - c) = \frac{1}{|\zeta - z|^3}\,\gamma_-(\zeta - z)\cdot\gamma_+(\zeta - c). \tag{3-2-2}$$

Proof. We suppose $c = 0$ and $r = 1$ for simplicity. A slight modification yields the general case. Let $|\zeta| = 1$ and $|z| < 1$. In the following we denote $a_1^p a_2^q$, for $a = (a_1, a_2)$, by $a^{(p,q)}$. We have:

$$\begin{aligned}2\pi^2\, A^+(z,\zeta) &= \sum_m \sum_{k=0}^{m+1}\sum_{l=0}^{m} \overline{\phi^{(m,l,k)}(\zeta)} \otimes \phi^{(m,l,k)}(z)\\ &= \sum_m \sum_{k=0}^{m+1}\sum_{l=0}^{m} \frac{(m+1-k)!}{k!l!(m-l)!}\\ &\qquad \begin{pmatrix} k\bar\epsilon_\zeta^{k-1}\bar\zeta^{(l,m-l)}\, k\epsilon_z^{k-1}z^{(l,m-l)} & -\bar\epsilon_\zeta^{k}\bar\zeta^{(l,m-l)}\, k\epsilon_z^{k-1}z^{(l,m-l)} \\ -k\bar\epsilon_\zeta^{k-1}\bar\zeta^{(l,m-l)}\, \epsilon_z^{k}z^{(l,m-l)} & \bar\epsilon_\zeta^{k}\bar\zeta^{(l,m-l)}\, \epsilon_z^{k}z^{(l,m-l)}\end{pmatrix}.\end{aligned}$$

We shall write

$$2\pi^2 A^+(z,\zeta) = \begin{pmatrix} A^{(1,1)} & A^{(1,2)} \\ A^{(2,1)} & A^{(2,2)} \end{pmatrix}.$$

We start from $A^{(2,2)}$.

$$\begin{aligned} A^{(2,2)} &= \sum_m \sum_{k=0}^{m+1} \sum_{l=0}^{m} \frac{(m+1-k)!}{k!l!(m-l)!} \,\overline{\epsilon}_\zeta^k \overline{\zeta}^{(l,m-l)} \,\epsilon_z^k z^{(l,m-l)} \\ &= \sum_m \sum_{k=0}^{m+1} \frac{(m+1-p)!}{k!m!} (\overline{\epsilon}_\zeta \epsilon_z)^k \,(\overline{\zeta}\cdot z)^m \\ &= \sum_{k=0} \sum_{m=k-1} \frac{(m+1-k)!}{k!m!} (\overline{\epsilon}_\zeta \epsilon_z)^k \,(\overline{\zeta}\cdot z)^m, \end{aligned}$$

here $\overline{\zeta}\cdot z = \overline{\zeta}_1 \cdot z_1 + \overline{\zeta}_2 \cdot z_2$.

Let B_k be the k-th summand. We have

$$\begin{aligned} B_0 &= \sum_{m=0} (m+1)(\overline{\zeta}\cdot z)^m = \frac{1}{(1-\overline{\zeta}\cdot z)^2} \\ B_1 &= \sum_{m=0} (\overline{\epsilon}_\zeta \epsilon_z)(\overline{\zeta}\cdot z)^m = \frac{\zeta\cdot\overline{z}}{(1-\overline{\zeta}\cdot z)^2} - 2\frac{|\zeta_1 z_2 - \zeta_2 z_1|^2}{(1-\overline{\zeta}\cdot z)^3} \end{aligned}$$

and

$$\begin{aligned} B_k &= \sum_{r=k-1} \frac{(m+1-k)!}{k!m!} (\overline{\epsilon}_\zeta \epsilon_z)^k (\overline{\zeta}\cdot z)^m \\ &= \frac{(-1)^{k-1}}{k!(k-2)!} (\overline{\epsilon}_\zeta \epsilon_z)^k \left\{(1-\overline{\zeta}\cdot z)^{k-2} \log(1-\overline{\zeta}\cdot z)\right\}, \quad \text{for } k \geq 2. \end{aligned}$$

Since the result is differentiated we do not care which branch of the logarithm we take. In fact, the summand B_k for $k \geq 2$ becomes as follows;

$$\begin{aligned} B_k &= \frac{1}{(1-\overline{\zeta}\cdot z)^2} (\zeta\cdot\overline{z})^k - 2k\frac{|\zeta_1 z_2 - \zeta_2 z_1|^2}{(1-\overline{\zeta}\cdot z)^3} (\zeta\cdot\overline{z})^{k-1} \\ &\quad + \frac{3}{2!} k(k-1) \frac{|\zeta_1 z_2 - \zeta_2 z_1|^4}{(1-\overline{\zeta}\cdot z)^4} (\zeta\cdot\overline{z})^{k-2} \\ &\quad - \frac{4}{3!} k(k-1)(k-2) \frac{|\zeta_1 z_2 - \zeta_2 z_1|^6}{(1-\overline{\zeta}\cdot z)^5} (\zeta\cdot\overline{z})^{k-3} + \cdots. \end{aligned}$$

Therefore we have

$$A^{(2,2)} = \sum_{k=0} \frac{1}{(1-\overline{\zeta}\cdot z)^2} (\zeta\cdot\overline{z})^k - \sum_{k=1} 2k\frac{|\zeta_1 z_2 - \zeta_2 z_1|^2}{(1-\overline{\zeta}\cdot z)^3} (\zeta\cdot\overline{z})^{k-1}$$

$$+\sum_{k=2}\frac{3}{2!}k(k-1)\frac{|\zeta_1 z_2-\zeta_2 z_1|^4}{(1-\overline{\zeta}\cdot z)^4}(\zeta\cdot\overline{z})^{k-2}-\cdots$$

$$=\frac{1}{(1-\overline{\zeta}\cdot z)^2}\frac{1}{(1-\zeta\cdot\overline{z})}-\frac{|\zeta_1 z_2-\zeta_2 z_1|^2}{(1-\overline{\zeta}\cdot z)^3}\frac{2}{(1-\zeta\cdot\overline{z})^2}$$

$$+\frac{|\zeta_1 z_2-\zeta_2 z_1|^4}{(1-\overline{\zeta}\cdot z)^4}\frac{3}{(1-\zeta\cdot\overline{z})^3}+\cdots$$

$$=\frac{(1-\zeta\cdot\overline{z})}{\left(1-\overline{\zeta}\cdot z-\zeta\cdot\overline{z}+|\zeta|^2|z|^2\right)^2}.$$

In the same way we have:

$$A^{(1,1)}=\sum_m\sum_{k=0}^{m+1}\sum_{l=0}^{m}\frac{(m+1-k)!}{k!l!(m-l)!}k\overline{\epsilon}_\zeta^{k-1}\overline{\zeta}^{(l,m-l)}\,k\epsilon_z^{k-1}z^{(l,m-l)}$$

$$=\frac{(1-\overline{\zeta}\cdot z)}{\left(1-\overline{\zeta}\cdot z-\zeta\cdot\overline{z}+|\zeta|^2|z|^2\right)^2}.$$

$$A^{(1,2)}=\sum_m\sum_{k=0}^{m+1}\sum_{l=0}^{m}\frac{(m+1-k)!}{k!l!(m-l)!}(-\overline{\epsilon}_\zeta^{k})\overline{\zeta}^{(l,m-l)}\,(k\epsilon_z^{k-1})z^{(l,m-l)}$$

$$=-\frac{-\zeta_1 z_2+\zeta_2 z_1}{\left(1-\overline{\zeta}\cdot z-\zeta\cdot\overline{z}+|\zeta|^2|z|^2\right)^2}.$$

$$A^{(2,1)}=\sum_m\sum_{k=0}^{m+1}\sum_{l=0}^{m}\frac{(m+1-k)!}{k!l!(m-l)!}(k\overline{\epsilon}_\zeta^{k-1})\overline{\zeta}^{(l,m-l)}\,(-\epsilon_z^{k})z^{(l,m-l)}$$

$$=\frac{\overline{\zeta_1 z_2-\zeta_2 z_1}}{\left(1-\overline{\zeta}\cdot z-\zeta\cdot\overline{z}+|\zeta|^2|z|^2\right)^2}.$$

Finally we have obtained the following expression for the kernel $A^+(z,\zeta)$:

$$2\pi^2A^+(z,\zeta)=\frac{1}{\left(1-\overline{\zeta}\cdot z-\zeta\cdot\overline{z}+|\zeta|^2|z|^2\right)^2}\begin{pmatrix}1-\overline{\zeta}\cdot z & -\zeta_1 z_2+\zeta_2 z_1\\ \overline{\zeta_1 z_2-\zeta_2 z_1} & 1-\zeta\cdot\overline{z}\end{pmatrix}$$

$$=\frac{1}{|z-\zeta|^4}\begin{pmatrix}\zeta_1-z_1 & \zeta_2-z_2\\ -\overline{\zeta}_2+\overline{z}_2 & \overline{\zeta}_1-\overline{z}_1\end{pmatrix}\cdot\gamma_+(\zeta)=K^\dagger(z,\zeta)\cdot\gamma_+(\zeta),$$

for $|\zeta|=1$ and $|z|<1$.

Let $A_c^-(z,\zeta)$ be the kernel function of A^-;

(3-2-3) $$A^-\phi(z)=\int_{B_r(c)}A_c^-(z,\zeta)\phi(\zeta)d\sigma(\zeta),\qquad z\in\mathbf{C}^2\setminus E_r(c).$$

THEOREM 3.4 (2). *For $|\zeta - c| = r$ and $|z - c| > r$ it holds that*

$$2\pi^2 A^-(z,\zeta) = -K^\dagger(z,\zeta)\cdot\gamma_+(\zeta - c). \tag{3-2-4}$$

PROOF. This time also we suppose $c = 0$ and $r = 1$. Let $|\zeta| = 1$ and $|z| > 1$.

$$\begin{aligned}2\pi^2 A^-(z,\zeta) &= \sum_m \sum_{k=0}^{m+1}\sum_{l=0}^{m} \overline{\phi^{-(m,l,k)}(\zeta)} \otimes \phi^{-(m,l,k)}(z)\\ &= \sum_m \sum_{k=0}^{m+1}\sum_{l=0}^{m} \frac{(m+1-k)!}{k!l!(m-l)!}\frac{1}{|z|^{2m+4}}\\ &\qquad \begin{pmatrix} \overline{\delta}_\zeta^k(\overline{\zeta_2 f(\zeta)})\,\delta_z^k(z_2 f(z)) & \overline{\delta}_\zeta^k(\zeta_1\overline{f(\zeta)})\,\delta_z^k(z_2 f(z)) \\ \overline{\delta}_\zeta^k(\overline{\zeta_2 f(\zeta)})\,\delta_z^k(\overline{z}_1 f(z)) & \overline{\delta}_\zeta^k(\zeta_1\overline{f(\zeta)})\,\delta_z^k(\overline{z}_1 f(z))\end{pmatrix},\end{aligned}$$

where we have put $f(z) = \overline{z}_1^l z_2^{m-l}$.

Let $d = (\overline{\delta}_\zeta\delta_z)^{k-1}$ and $g = \overline{f}(\zeta)f(z)$. Then $2\pi^2 A^-(z,\zeta)$ is written as:

$$\begin{aligned}&2\pi^2 A^-(z,\zeta)\\ &= \sum_m \sum_{k=0}^{m+1}\sum_{l=0}^{m} \frac{(m+1-k)!}{k!l!(m-l)!}|z|^{-2m-3}\gamma_-(z)\begin{pmatrix} p^2 dg & -p\overline{\delta}_\zeta dg \\ -pd\delta_z g & d\overline{\delta}_\zeta\delta_z g\end{pmatrix}\gamma_+(\zeta).\end{aligned}$$

Each entry is calculated as before. The sum of the series in (2,2) component, for example, becomes

$$\begin{aligned}\sum_m \sum_{k=0}^{m+1}\sum_{l=0}^{m} &\frac{(m+1-k)!}{k!l!(m-l)!}|z|^{-2m-3}(\overline{\delta}_\zeta\delta_z)^k(\zeta_1\overline{z}_1)^l(\overline{\zeta}_2 z_2)^{m-l}\\ &= \frac{\left(|z|^2 - (\overline{\zeta}_1 z_1 + \zeta_2\overline{z}_2)\right)}{|z|\,|z-\zeta|^4}.\end{aligned}$$

We have thus

$$\begin{aligned}2\pi^2 A^-(z,\zeta) &= \frac{1}{|z-\zeta|^4}\gamma_-(z)\gamma_+(z)\begin{pmatrix} z_1-\zeta_1 & z_2-\zeta_2 \\ -\overline{z}_2+\overline{\zeta}_2 & \overline{z}_1-\overline{\zeta}_1\end{pmatrix}\gamma_+(\zeta)\\ &= -K^\dagger(z,\zeta)\gamma_+(\zeta).\end{aligned}$$

Theorem is proved.

From Propositions 2.3, 2.4, and Theorem 3.4 we have the following theorem on a Taylor expansion.

THEOREM 3.5 (Taylor expansion).

(1) *Let* $\varphi \in C^\infty(G, S^+)$ *be a zero mode spinor on a domain* G *and let* $c \in G$. *Then we have an expansion*

$$\varphi(z) = \sum_{m,l,k} C^{(m,l,k)}\, \phi^{(m,l,k)}(z-c),$$

in a ball $\{|z-c| \le r\} \subset G$, *where* $C^{(m,l,k)}$ *is given by*

$$C^{(m,l,k)} = \frac{1}{2\pi^2}\int_{B_r(c)} \langle \gamma_+(\zeta - c)\varphi(\zeta)\,,\, \psi^{(m,l,k)}(\zeta - c)\rangle\, \sigma(d\zeta).$$

(2) *Let* $\varphi \in \mathcal{N}(\mathrm{C}^2 \setminus K, S^+)$ *be a zero mode spinor on the compliment of a compact set* $K \subset \{|z| \le r\}$. *Then we have an expansion*

$$\varphi(z) = \sum_{m,l,k} C^{-(m,l,k)}\, \phi^{-(m,l,k)}(z),$$

on $\{|z| > r\}$, *where*

$$C^{-(m,l,k)} = \frac{1}{2\pi^2}\int_{B_r} \langle \gamma_+(\zeta)\varphi(\zeta)\,,\, \psi^{-(m,l,k)}(\zeta)\rangle\, \sigma(d\zeta).$$

The expansion of Cauchy kernel and a Taylor expansion as well as the Laurent expansion in R^{m+1} was obtained earlier by Delanghe and Sudbery using their basis of spherical monogenics mentionned at the end of Section 1.5. See, for example, Sections 11 and 12 of [B-D-F], Chapter II of [D-S-Sč] and Theorems 10 and 11 of [S]. Taylor and Laurent series of monogenic functions on a spherical domain were studied in [So-2, V-L].

4. Meromorphic spinors and their residues

4.1. Expansion of a zero mode spinor on an annulus in a series of $\phi^{\pm(m,l,k)}$

THEOREM 4.1. *Let* φ *be a smooth even spinor on* $0 \le r < |z-c| < R \le \infty$ *such that* $D\varphi = 0$. *Then we have the expansion*

$$\begin{aligned}\varphi(z) &= \sum_{m,l,k} C_{(m,l,k)}\, \phi^{(m,l,k)}(z-c) \\ &\quad + \sum_{m,l,k} C_{-(m,l,k)}\, \phi^{-(m,l,k)}(z-c), \qquad r < |z-c| < R.\end{aligned} \tag{4-1-1}$$

The coefficients are uniquely determined by (m,l,k) *and* c, *and are given by*

$$C_{\pm(m,l,k)} = \frac{1}{2\pi^2}\int_{B_\rho(c)} \langle \gamma_+(\zeta - c)\varphi(\zeta)\,,\,\psi^{\pm(m,l,k)}(\zeta - c)\rangle\,\sigma(d\zeta) \tag{4-1-2}$$

for any ρ *such that* $r < \rho < R$.

PROOF. Let s, t be such that $r < s < t < R$. From the integration formula (Proposition 2.3) applied to the domain $s < |z-c| < t$ we have

$$\begin{aligned}\varphi(z) &= \frac{1}{2\pi^2}\int_{B_t(c)} K^\dagger(z,\zeta)\gamma_+(\zeta - c)\varphi(\zeta)\sigma(d\zeta)\\ &\quad - \frac{1}{2\pi^2}\int_{B_s(c)} K^\dagger(z,\zeta)\gamma_+(\zeta - c)\varphi(\zeta)\sigma(d\zeta), \quad s < |z-c| < t.\end{aligned}$$

It follows from Theorem 3.4 (1) and (2) that;

$$\varphi(z) = \sum_{m,l,k} C^{(m,l,k)}_{(c,t)}\,\phi^{(m,l,k)}(z-c) + \sum_{m,l,k} C^{-(m,l,k)}_{(c,s)}\,\phi^{-(m,l,k)}(z-c).$$

Now from Lemma 3.3 $C^{(m,l,k)}_{(c,t)}$ is given by

$$C^{(m,l,k)}_{(c,t)} = \frac{1}{2\pi^2}\int_{B_t(c)} \langle \gamma_+(\zeta - c)\varphi(\zeta)\,,\,\psi^{(m,l,k)}(\zeta - c)\rangle\,\sigma(d\zeta).$$

But, since $D\varphi = 0$ and $D^\dagger\psi_c^{(m,l,k)} = 0$ on $r < |\zeta - c| < R$, we find that it is independent of t. $C^{-(m,l,k)}_{(c,s)}$ is also independent of s.

4.2. Given a zero mode spinor φ on $0 < |z-c| < r$;

$$\varphi \in C^\infty(E_r(c)\setminus\{c\}, S^+), \quad D\varphi = 0.$$

In the expansion of φ in (4-1-1) the second part

$$\sum_{m,l,k} C_{-(m,l,k)}\,\phi_c^{-(m,l,k)}$$

is called the *principal part of* φ *at* c.

DEFINITION. Let G be a domain in $\mathbf{C}^2$ and let E be a discrete subset of G. A zero mode spinor φ on $G\setminus E$ is said to be meromorphic on G if its principal part has only finitely many terms at every point of E. A point of E is called a pole of φ.

Let φ be a meromorphic spinor on a domain G and let

$$\sum_{m,l,k} C_{-(m,l,k)}\phi_c^{-(m,l,k)}$$

be its principal part at a point $c \in G$. We call the vector

$$\begin{pmatrix} -C_{-(0,0,1)} \\ C_{-(0,0,0)} \end{pmatrix}, \tag{4-2-1}$$

the *residue* at $z = c$ of φ and we denote it by

$$\operatorname{Res}\varphi(c). \tag{4-2-2}$$

$\operatorname{Res}\varphi(c)$ are the coefficients of the term of order $O\left(\frac{1}{|z-c|^3}\right)$.

PROPOSITION 4.2.

$$\operatorname{Res}\varphi(c) = \frac{1}{2\pi^2}\int_{B_\epsilon(c)} \gamma(z-c)\varphi(z)\sigma(dz), \tag{4-2-3}$$

for any sufficiently small $\epsilon > 0$.

PROOF. Since

$$\psi^{-(0,0,0)}(z-c) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \psi^{-(0,0,1)}(z-c) = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

we have

$$\begin{aligned} C_{-(0,0,0)} &= \frac{1}{2\pi^2}\int_{B_\epsilon(c)} \langle \gamma_+(z-c)\varphi(z), \psi^{-(0,0,0)}(z-c)\rangle\,\sigma(dz) \\ &= \frac{1}{2\pi^2}\int_{B_\epsilon(c)} (\gamma_+(z-c)\varphi)_2\sigma(dz). \end{aligned}$$

Similarly

$$C_{-(0,0,1)} = -\frac{1}{2\pi^2}\int_{B_\epsilon(c)} (\gamma_+(z-c)\varphi)_1\sigma(dz).$$

Here $(\phi)_j$ is the j-th component of the spinor ϕ.

THEOREM 4.3. *Let* φ *be a meromorphic spinor on a domain* $G' \subset \mathbf{C}^2$ *and let* $E = \{p_1, p_2, \ldots, p_n\}$ *be the poles of* φ. *Suppose* E *is contained in a relatively compact subdomain* $G \subset G'$ *with smooth boundary* ∂G. *Then we have*

$$\frac{1}{2\pi^2}\int_{\partial G} (\gamma_{\partial G}\,\varphi)\,d\sigma = \sum_{i=1}^{n} \operatorname{Res}\varphi(p_i). \tag{4-2-4}$$

REMARK. Residue theorem for (left) monogenic functions at the singular point were given in Theorem 12.4.3 of [B-D-S]. Since their basis of expansion, (1-5-5) and (1-5-6), is different from ours their definition and ours are different, but the values of both residues of a meromorphic spinor coincide of course. On the other hand the residue of monogenic forms discussed in [D-S-Sč, Sč] is related to the higher dimensional analogue of $\overline{\partial}$ operator and residue formula there was given by cohomological formula.

4.3. Here we study the effect of conformal transformations on the system $\{\phi^{\pm(m,l,k)}\}$. Let $f : U \longrightarrow \mathrm{R}^4$ be a conformal transformation. Then the map f induces a Spin(4)-equivariant map $f_\flat$ of Spin(4)-principal bundles and it yields a bundle isometry $f' = \Delta(f_\flat) : S \longrightarrow S'$ of the associated spinor bundles. The Dirac operator is conformally covariant, that is, if D' is the Dirac operator corresponding to the metric g'; $f^*g' = e^{2u}g$, then

$$D'_{f(z)} = F \cdot D_z \cdot F^{-1}, \tag{4-3-1}$$

where u is a smooth function on U and $F = e^{-\frac{3}{2}u}f'$, [Hi] [L-M].

Under the conformal transformations the vector fields θ_0, ϵ and $\overline{\epsilon}$ do not change their forms, while the radial vector field $\frac{\partial}{\partial n}$ does not change its form under the translations, orthogonal transformations and dilations, but under Kelvin inversion it changes the sign. For Kelvin inversion, we have

$$F \cdot \frac{\partial}{\partial n_z} \cdot F^{-1} = \frac{\partial}{\partial n_w} - \frac{3}{2}|w|.$$

Each vector field θ_0, ϵ and $\overline{\epsilon}$ annhilates the conformal factor u.

We note also that the chiral matrix changes according to

$$(f')^{-1}\gamma f' = e^{-u}\gamma.$$

Now let U be a domain containing the disk $\{|z| \leq 1\}$. From what we have verified above we can verify

$$\partial\!\!\!/_{f(z)} = \pm F\, \partial\!\!\!/_z\, F^{-1} = \pm f'\, \partial\!\!\!/_z\, (f')^{-1}, \quad \text{on } |z| = 1. \tag{4-3-2}$$

Hence on the sphere $f(\{|z| = 1\}) \subset f(U)$ the eigenvalues of $\partial\!\!\!/$ are $-\frac{n+3}{2}$, $\frac{n}{2}$, $n = 0, \pm1, \dots$ if f is orientation preserving, while they changes to $\frac{n+3}{2}$, $-\frac{n}{2}$, $n = 0, \pm1, \dots$ if f is orientation reversing. The corresponding eigenspinors become $f'\phi^{\pm(m,l,k)}$, that are extended as zero mode spinors to $\mathrm{R}^4 \setminus f(0)$ by $F\phi^{\pm(m,l,k)}$. In particular by a coordinate change $T \in SO(4)$ we have the same eigenvalues of $\partial\!\!\!/$ and the eigenspinors are given by $\phi^{\pm(m,l,k)} \circ T$, and our theory is independent of the chosen complex structure $\mathrm{C}^2 \simeq \mathrm{R}^4$. By the transformation $f(z) = c + rz$, $r > 0$, we find that the eigenspinors on $|z - c| = r$ are given by

$$r^{\mp(m+\frac{3}{2})}\phi^{\pm(m,l,k)}(z - c).$$

We have used it in Section 3.1. We shall see soon the effect of inversion: $f(z) = \frac{\bar{z}}{|z|^2}$. Having verified the covariance of our theory on R^4 under conformal transformations we can extend it to a theory on a manifold which is locally R^4 and patched together by conformal transformations, that is, on a conformally flat 4-manifold.

Now we shall see the theory on S^4 in more detail and discuss the expansion. S^4 is obtained by patching up two copies of C^2 together by the inversion $w = v(z) = \frac{\bar{z}}{|z|^2}$. We shall denote these two local coordinates by C^2 and $\widehat{\mathrm{C}}^2$. v has the conformal weight $u = -\log|z|^2$. The surface measure on $\{|w| = \frac{1}{R}\} = \{|z| = R\}$ changes according to $\sigma(dw) = -\frac{1}{R^6}\sigma(dz)$.

The even (resp. odd) spinor bundle $S^\pm$ on M is obtained by the identification

$$\text{(4-3-3)} \qquad \mathrm{C}^2 \times \Delta^\pm \ni (z, \xi) \longleftrightarrow \left(w = v(z),\ \hat{\xi}(w) = |w|^{-3}\overline{\gamma_\pm \xi}(v^{-1}w)\right) \in \widehat{\mathrm{C}}^2 \times \Delta^\mp.$$

Thus an even spinor ϕ on a subset U of M is a pair of $\phi \in C^\infty(U \cap \mathrm{C}^2 \times \Delta^+)$ and $\widehat{\phi} \in C^\infty(U \cap \widehat{\mathrm{C}}^2 \times \Delta^-)$ such that $\widehat{\phi}(w) = \overline{|z|^3(\gamma_+ \cdot \phi)(z)}$ for $w = v(z)$.

The Cauchy integral on $\widehat{\mathrm{C}}^2$ has the same matrix representation as that on C^2, for example,

$$\text{(4-3-4)} \qquad \widehat{K}^\dagger(w, \eta) = \frac{1}{|\eta - w|^3}\gamma_-(\eta - w) = \frac{1}{|\eta - w|^4}\begin{pmatrix} \overline{\eta}_1 - \overline{w}_1 & -\eta_2 + w_2 \\ \overline{\eta}_2 - \overline{w}_2 & \eta_1 - w_1 \end{pmatrix}.$$

We can verify by a direct calculation that

$$\text{(4-3-5)} \qquad \widehat{K}^\dagger\widehat{\phi} = \widehat{K^\dagger\phi}.$$

Proposition 4.4. *Let G be a domain in $\widehat{\mathrm{C}}^2$ that is a neighborhood of* 0. *Let $\widehat{\varphi} \in C^\infty(G, S^+) = C^\infty(G, \Delta^-)$. We suppose that $\widehat{D}\widehat{\varphi} = 0$. Then we have the integral representation*

$$\text{(4-3-6)} \qquad \widehat{\varphi}(w) = \frac{1}{2\pi^2}\int_{\partial G}\widehat{K}^\dagger(w, \eta)(\gamma_+\widehat{\varphi})(\eta)\sigma(d\eta).$$

Now we shall investigate the Laurent expansion on a neighborhood of the point at infinity. First we note that the following relations hold:

$$\text{(4-3-7)} \qquad \begin{aligned} \widehat{\phi}^{(m,l,k)}(w) &= \psi^{(m,l,k)}(w), \qquad w \in \widehat{\mathrm{C}}^2 \setminus 0, \\ \widehat{\phi}^{-(m,l,k)}(w) &= \psi^{-(m,l,k)}(w), \qquad w \in \widehat{\mathrm{C}}^2 \setminus 0, \end{aligned}$$

where $\psi^{\pm(m,l,k)}$ is given by the formula (1-5-1). This implies $\widehat{\phi}^{-(m,l,k)} \sim 0(|w|^m)$. Hence we define, if $m \geq 1$, $\widehat{\phi}^{-(m,l,k)}$ to be 0 at $w=0$, while $\widehat{\phi}^{-(0,0,k)}$, $k=0,1$, being constants, are extended to $w=0$ naturally.

From (4-3-7) and (1-5-4) we have

$$\begin{aligned} &\widehat{D}\,\widehat{\phi}^{-(m,l,k)}(w)=0,\\ &\widehat{D}\,\widehat{\phi}^{(m,l,k)}(w)=0, \quad w\neq 0. \end{aligned} \tag{4-3-8}$$

The relations (4-3-5), (4-3-7) and Theorem 3.4 yield the following:

PROPOSITION 4.5.

$$\widehat{K}^{\dagger}(w,\eta)\,\gamma_{+}(\eta)=2\pi^2\widehat{A}^{-}(w,\eta)=\sum_{m}\sum_{k=0}^{m+1}\sum_{l=0}^{m}\overline{\psi^{-(m,l,k)}(\eta)}\otimes\psi^{-(m,l,k)}(w), \tag{4-3-9}$$

for $|\eta|=\frac{1}{R}$ *and* $|w|<\frac{1}{R}$. *Similarly*

$$-\widehat{K}^{\dagger}(w,\eta)\gamma_{+}(\eta)=\sum_{m}\sum_{k=0}^{m+1}\sum_{l=0}^{m}\overline{\psi^{(m,l,k)}(\eta)}\otimes\psi^{(m,l,k)}(w), \tag{4-3-10}$$

for $|\eta|=\frac{1}{R}$ *and* $|w|>\frac{1}{R}$.

Let $\widehat{E}_r(0)=\{|w|<r\}\subset\widehat{C}^2$ and let

$$\mathcal{N}(\widehat{E}_r(0))=\{\widehat{\phi}\in H^1(\widehat{E}_r(0),S^{+})=H^1(\widehat{E}_r(0),\Delta^{-})\,;\,\widehat{D}\widehat{\phi}=0\}. \tag{4-3-11}$$

Then we have an obvious isomorphism:

$$\mathcal{N}(C^2\setminus E_R(0))\simeq\mathcal{N}(\widehat{E}_{\frac{1}{R}}(0)). \tag{4-3-12}$$

By this isomorphism we see that the assertion of Proposition 4.4 is the same as that of Proposition 2.4.

Let φ be a zero mode spinor on $0<|w|<\frac{1}{R}$. We can prove by the same argument as in 4.1 the Laurent expansion of φ around $w=0$. By the isomorphism (4-3-12) it translates to the Laurent expansion around the point at infinity of C^2;

THEOREM 4.6. *Let φ be a smooth even spinor on $\{z;\,R<|z|\}$. Suppose that $D\varphi=0$. Then we have the following expansion at infinity*:

(4-3-13)

$$\varphi(z)=\sum_{(m,l,k)}\widehat{B}_{-(m,l,k)}\,\phi^{(m,l,k)}(z)+\sum_{(m,l,k)}\widehat{B}_{(m,l,k)}\,\phi^{-(m,l,k)}(z), \quad R<|z|.$$

The coefficients are given by

$$\widehat{B}_{\mp(m,l,k)} = -\frac{1}{2\pi^2}\int_{B_\rho(0)} \langle \gamma_+\varphi(z), \psi^{\mp(m,l,k)}(z)\rangle\, d\sigma(z) \tag{4-3-14}$$

for any ρ *such that* $\rho > \frac{1}{R}$.

4.4. Let φ be a zero mode spinor on $|z| > R$. We say that φ is meromorphic at infinity if, in its Laurent expansion at infinity, there are only finitely many $\widehat{B}_{-(m,l,k)}$'s that do not vanish. We call the vector

$$\begin{pmatrix} B_{(0,0,1)} \\ -B_{(0,0,0)} \end{pmatrix}, \tag{4-4-1}$$

the *residue* at infinity of φ and we denote it by

$$\operatorname{Res}\varphi(\infty). \tag{4-4-2}$$

$\operatorname{Res}\varphi(\infty)$ is the coefficients of the term of order $O(|z|^3)$.

Lemma 4.7.

$$\operatorname{Res}\varphi(\infty) = -\frac{1}{2\pi^2}\int_{B_\rho(0)} \gamma_+\varphi(z)d\sigma(z) \quad \text{for any } \rho > R. \tag{4-4-3}$$

Finally we have the residue theorem.

Theorem 4.8. *Let* φ *be a meromorphic spinor on* S^4 *with poles at the points* $c_1, c_2, \ldots, c_m \in S^4$. *Then*

$$\sum_{k=1}^{m} \operatorname{Res}\varphi(c_k) = 0. \tag{4-4-4}$$

The theorem follows from (4-2-3) and (4-4-3).

5. Residue theorem on a conformally flat manifold

5.1. Conformally flat manifolds

Let M be a riemannian 4-manifold with conformally flat metric. A Riemannian manifold is conformally flat if and only if its Weyl tensor vanishes. We know that when M is compact and simply connected M is conformally equivalent to S^4.

We suppose moreover that M has a spin structure and we fix it.

There exists a system of coordinate neighborhoods $(U_\lambda, \chi_\lambda)$ in M such that each $G_\lambda = \chi_\lambda(U_\lambda)$ is a domain in C^2 with local coordinate $\{z_1^\lambda, z_2^\lambda\}$ and such that the transition function $f_{\mu\lambda} = \chi_\mu \chi_\lambda^{-1}$ is a conformal transformation;

$$f_{\mu\lambda}^* \left(dz_1^\mu d\overline{z}_1^\mu + dz_2^\mu d\overline{z}_2^\mu\right) = e^{2u_{\mu\lambda}} \left(dz_1^\lambda d\overline{z}_1^\lambda + dz_2^\lambda d\overline{z}_2^\lambda\right), \tag{5-1-1}$$

where $u_{\mu\lambda}$ is a smooth function on $G_\mu \cap G_\lambda$.

A meromorphic spinor on M is, by definition, a smooth spinor φ on $M \setminus E$, E being a discrete subset, such that, for each λ, $\varphi_\lambda = (\chi_\lambda)'\varphi$ is a meromorphic spinor on $G_\lambda \subset \mathrm{C}^2$ with poles at $G_\lambda \cap \chi_\lambda(E)$. This is equivalent to say that the family of meromorphic spinors φ_λ on G_λ such that $\varphi_\mu = (f_{\mu\lambda})'\varphi_\lambda$ defines a meromorphic spinor on M.

Let φ be a meromorphic spinor with a pole at $z \in G \subset \mathrm{C}^2$. Let f be a conformal transformation defined on G. If f is either a translation, an orthogonal transformation or a dilation, the coefficients of the Laurent expansion at $f(z)$ of $f'\varphi$ do not change. While by the inversion which sends z to the point at infinity, the coefficient $B_{\mp(m,l,k)}$ of the expansion of $f'\varphi$ at infinity is equal to the minus of $C_{\pm(m,l,k)}$ as we have seen in 4.3 and 4.4. Thus, for a meromorphic spinor on M, the order of pole and the residue are defined independently of the local coordinates.

THEOREM 5.1. *Suppose M is compact and let $E = \{c_i\}_i$ be a finite set of points. We suppose that there exist spinors $\psi_k \in C^\infty(M, S^-)$, $k = 0, 1$, such that $D^\dagger \psi_k = 0$ and such that $\psi_k = \psi_{c_i}^{-(0,0,k)}$ in a neighborhood of c_i for any $c_i \in E$. Let φ be a meromorphic spinor with poles at E. Then*

$$\sum_i \operatorname{Res} \varphi(c_i) = 0. \tag{5-1-2}$$

PROOF. Let B_i be a small ball in $U(c_i)$.

The k-th component of

$$\operatorname{Res} \varphi(c_i) = \frac{1}{2\pi^2} \int_{\partial B_i} (\gamma_{\partial B_i} \varphi)_k \, d\sigma = \frac{1}{2\pi^2} \int_{\partial B_i} \langle \gamma_{\partial B_i} \varphi, \psi_k \rangle \, d\sigma,$$

for $k = 0, 1$. Stokes' formula yields that

$$\sum_i \frac{1}{2\pi^2} \int_{\partial B_i} \langle \gamma_{\partial B_i} \varphi, \psi_k \rangle \, d\sigma = 0.$$

Therefore

$$\sum_i \operatorname{Res} \varphi(c_i) = 0.$$

REMARK. Let $\mathcal{N}^\dagger$ be the sheaf (of linear spaces) of germs of zero mode odd spinors. Let $\mathcal{L}_E$ be the subsheaf of those spinors that vanish at E. We have the following short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{L}_E \longrightarrow \mathcal{N}^\dagger \longrightarrow \mathcal{N}^\dagger/\mathcal{L}_E \longrightarrow 0.$$

If E is a point c then $\mathcal{L}_c$ is the sheaf of linear spaces spanned by the germs $\psi_c^{-(m,l,k)}$, $m \geq 1$ and $\mathcal{N}^\dagger/\mathcal{L}_c$ is generated by $\psi_c^{-(0,0,k)}$, $k = 0, 1$. The hypothesis in Theorem 5.1 can be stated as

$$H^1(M, \mathcal{L}_E) = 0. \tag{5-1-3}$$

In our forthcoming paper, [K-3], we shall introduce, for a meromorphic spinor on M, a concept analogous to a divisor of a meromorphic function on a Riemann surface, and study cohomology groups of zero mode spinors with poles and zeros at a given divisor.

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DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCE AND ENGINEERING
UNIVERSITY OF WASEDA
3-4-1 OKUBO, SHINJUKU-KU
TOKYO, JAPAN
E-mail: kori@waseda.jp