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Cohomology Groups of Harmonic Spinors on Conformally Flat Manifolds

Tosiaki Kori

Abstract. We shall investigate various properties of the sheaf of harmonic spinors $\mathcal{N}$ on C^2 and, more generally, on conformally flat spin 4-manifolds. We prove the Runge approximation theorem on C^2, and the vanishing of cohomologies; $H^1(\mathrm{C}^2, \mathcal{N}) = 0$ and $H^1(S^4, \mathcal{N}) = 0$. We shall introduce a concept of divisors of meromorphic spinors on a compact conformally flat spin 4-manifold , and give an analogy of Riemann-Roch theorem.

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1. Introduction

This is a continuation of our previous research on harmonic spinors on conformally flat spin 4-manifolds, [8], [9]. Let (M, S) be a 4-dimensional Riemannian manifold with a compatible spin structure, $S = S^+ \oplus S^-$, $S^\pm$ being the two spin bundles associated to two half-spin representations of $Spin(4)$. Let $D : S^+ \longrightarrow S^-$ be the (half) Dirac operator. A spinor $\varphi \in C^\infty(S^+)$ is called a *harmonic spinor* if $D\varphi = 0$. Let $\mathcal{N}$ be the sheaf of harmonic spinors. Since D is an elliptic operator, we have the exact sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{S}^+ \xrightarrow{D} \mathcal{S}^- \longrightarrow 0, \tag{1.1}$$

where $\mathcal{S}^\pm$ is the sheaf of even (resp. odd) spinors. Therefore

$$\begin{aligned} H^p(M, \mathcal{N}) &= 0 \quad \text{for } p \geq 2, \\ H^1(M, \mathcal{N}) &= coker\{D; C^\infty(M, \mathcal{S}^+) \longrightarrow C^\infty(M, \mathcal{S}^-)\}, \\ H^0(M, \mathcal{N}) &= ker\{D; C^\infty(M, \mathcal{S}^+) \longrightarrow C^\infty(M, \mathcal{S}^-)\}. \end{aligned} \tag{1.2}$$

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$H^1(M,\mathcal{N})$ and $H^0(M,\mathcal{N})$ are endowed with the structure of Frechet spaces. When M is compact these are finite dimensional, and we have

$$\dim H^0(M,\mathcal{N}) - \dim H^1(M,\mathcal{N}) = \mathrm{index}(D) = -\frac{1}{24}p_1(M), \tag{1.3}$$

from the Atiyah-Singer index theorem, where $p_1(M)$ is the first Pontryagin number of the 4-manifold M.

We are interested in the vanishing of $H^1(M,\mathcal{N})$. We shall prove it if M is an open subset of C^2 and for $M = S^4$. We conjecture the vanishing of $H^1(M,\mathcal{N})$ for any non-compact conformally flat spin 4-manifold.

Now we shall explain the contents of each sections After the introduction of ingredients on spinor analysis that are studied in [8], we shall prove in Section 3 the Runge approximation theorem, that is, any harmonic spinor on a compact subset K of a domain $G \subset \mathrm{C}^2$ such that $G \setminus K$ has no relatively compact connected component can be approximated uniformly on K by harmonic spinors on G. Several Runge type theorems for Clifford algebra valued functions on a domain in R^n, as well as for Clifford modules, were proved earlier in [2, 3, 11]. We think our argument, which is close to that by Hörmander [7], is worth to be presented. In 3.2 we prove that the Dirac equation $D\varphi = \psi$ has a solution on any open subset G of C^2. Hence we have

$$H^1(G,\mathcal{N}) = 0. \tag{1.4}$$

We can verify the covariance of our theory under conformal transformations on R^4, thus we can extend it to a conformally flat 4-manifold. In particular the above stated properties are independent of the complex structure on R^4. In Section 4 we shall deal with the cohomology of $\mathcal{N}$ on a conformally flat 4-manifold M. We shall see that the cohomology group $H^1(M,\mathcal{N})$ is calculated by a Leray covering. Thus we see that the well known argument to have the classical result $H^1(P^1(\mathrm{C}),\mathcal{O}) = 0$ is valid to prove

$$H^1(S^4,\mathcal{N}) = 0. \tag{1.5}$$

We shall prove that the restriction map

$$H^1(G',\mathcal{N}) \longrightarrow H^1(G,\mathcal{N})$$

has a finite dimensional image for a relatively compact open subset G of an open subset $G' \subset M$. This implies the existence of a non-trivial meromorphic spinor on a relatively compact open subset of M.

Our results should extend to even dimensional conformally flat manifolds. In fact we have already the Runge approximation theorem on a domain in R^n as was proved earlier in [3], so if we adopted it we could in principle obtain the results in Section 4 also on a domain in R^{2n}, but it would be very complicated to write it down because of the 2^n components of spinors.

In Section 5 we shall study the cohomology groups of meromorphic spinors, that is, harmonic spinors with singularity, on compact conformally flat 4-manifolds. We develop a divisor theory for meromorphic spinors. But, because we have no product operation on spinors, this is a linear analogy of classical divisors of meromorphic functions. We shall show a Riemann-Roch type theorem for the cohomological dimensions of meromorphic spinors with prescribed divisor. More precisely, let E be a divisor on a compact conformally flat manifold M and let $\mathcal{L}_E$ be the sheaf of meromorphic spinors having the poles at E of order less than $\deg E$. Then we have

$$\dim H^0(M, \mathcal{L}_E) - \dim H^1(M, \mathcal{L}_E) = \deg E, \tag{1.6}$$

$$H^p(M, \mathcal{L}_E) = 0 \text{ for } p \geq 2. \tag{1.7}$$

2. Preliminaries on the Dirac operator and the Cauchy kernel

Here we shall summarize some ingredients of spinor analysis that are now well known, [1, 2, 10].

2.1. Dirac operator and harmonic spinors

Let $\Delta = \Delta^+ \oplus \Delta^-$ be an irreducible complex representation of the Clifford algebra $Clif(\mathrm{R}^4)$; $Clif(\mathrm{R}^4) \otimes \mathrm{C} = End(\Delta)$. Δ decomposes to irreducible representations $\Delta^\pm$ of $Spin(4)$, each of which has $\dim \Delta^\pm = 2$. Let $S = \mathrm{R}^4 \times \Delta$ be the spinor bundle on R^4. The corresponding bundle S^+ (resp. S^-) is called even (resp. odd) spinor bundle.

We shall choose complex coordinates and look at $\mathrm{R}^4 \simeq \mathrm{C}^2$. Our theory does not depend on the complex structure but on the conformal structure on the manifold. The complex coordinates description is adopted for convenience of notation, though this notation allows us to see things in perspective, for example, our formulas for eigenspinors of the Dirac operator fit for the $SU(2)$ representation theory.

Given a smooth boundary ∂G of a region G we shall denote by $\gamma_{\partial G}$ the Clifford multiplication of the outer unit normal on ∂G. We shall abbreviate it as γ if it is obvious which boundary we are considering. $\gamma_{\partial G}$ changes the chirality:

$$\gamma_{\partial G} : S^+ \oplus S^- \longrightarrow S^- \oplus S^+; \qquad \gamma_{\partial G}^2 = 1. \tag{2.1}$$

Let γ_0 denote the Clifford multiplication of the radial vector $\frac{\partial}{\partial n}$, the unit normal to the unit sphere. The chiral decomposition of γ_0 becomes

$$\gamma_0 = \begin{pmatrix} 0 & \gamma_- \\ \gamma_+ & 0 \end{pmatrix} : \begin{matrix} S^+ \\ \oplus \\ S^- \end{matrix} \longrightarrow \begin{matrix} S^+ \\ \oplus \\ S^- \end{matrix} . \tag{2.2}$$

The Dirac operator is defined by

$$\mathcal{D} = c \circ d, \tag{2.3}$$

where $d : S \longrightarrow S \otimes T^*\mathrm{C}^2 \simeq S \otimes T\mathrm{C}^2$ is the exterior differentiation and $c : S \otimes T\mathrm{C}^2 \longrightarrow S$ is the bundle homomorphism coming from the Clifford multiplication. $\mathcal{D}$ is an elliptic operator.

By means of the decomposition $S = S^+ \oplus S^-$, the Dirac operator is decomposed into chiral components:

$$\mathcal{D} = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} : C^\infty\left(\mathrm{C}^2, S^+ \oplus S^-\right) \longrightarrow C^\infty\left(\mathrm{C}^2, S^+ \oplus S^-\right). \tag{2.4}$$

An even (resp. odd) spinor φ is called *harmonic spinor* if $D\varphi = 0$ (resp. $D^\dagger\varphi = 0$). We denote by $\mathcal{N}(U)$ (resp. $\mathcal{N}^\dagger(U)$) the set of even (resp. odd) harmonic spinors on an open set U.

Remark 2.1. In [9] we called $\varphi \in \mathcal{N}(M)$ a *zero mode* spinor on M. The reason why we preferred it was that, on a noncompact manifold M, the condition of harmonicity, $D^\dagger D\varphi = 0$, is not equivalent to $D\varphi = 0$.

The following fundamental properties of harmonic spinors are well known [1].

Theorem 2.2. *A harmonic spinor on a connected open set vanishes identically if it vanishes on an open subset.*

Theorem 2.3. *If U and V are domains in C^2 such that $\overline{V}$ is compact in U, then the restriction map $r^U_V : \mathcal{N}(U) \longrightarrow \mathcal{N}(V)$ is compact.*

The following Stokes' formula holds for $\phi \in C^\infty(\overline{G}, S^+)$ and $\psi \in C^\infty(\overline{G}, S^-)$:

$$\int_G < D\phi, \psi > dV + \int_G < \phi, D^\dagger\psi > dV = \int_{\partial G} < \gamma\,\phi, \psi > d\sigma. \tag{2.5}$$

We shall denote in the sequel

$$(\varphi_1, \varphi_2) = \int_{\mathrm{C}^2} < \varphi_1, \varphi_2 > dV, \qquad \text{for } \varphi_1, \varphi_2 \in C^\infty(\mathrm{C}^2, S). \tag{2.6}$$

2.2. Cauchy integral formula

The Cauchy kernel is the Clifford multiplication of the radial component of the gradient of the Newton kernel, [2, 3, 8]. In our description it is defined as follows. We put, for any pair $\zeta \neq z$,

$$\mathcal{K} = \frac{1}{|\zeta - z|^3}\gamma_0(\zeta - z) \,: C^\infty(\mathrm{C}^2, S) \longrightarrow C^\infty(\mathrm{C}^2, S). \tag{2.7}$$

$\mathcal{K}$ decomposes after $S^+ \oplus S^-$ as

$$\mathcal{K} = \begin{pmatrix} 0 & K^\dagger \\ K & 0 \end{pmatrix}, \tag{2.8}$$

$$K^\dagger(z, \zeta) = \frac{1}{|\zeta - z|^3}\gamma_-(\zeta - z), \qquad K(z, \zeta) = \frac{1}{|\zeta - z|^3}\gamma_+(\zeta - z). \tag{2.9}$$

Proposition 2.4.

$$D_z K^\dagger(z, \zeta) = 0, \qquad D^\dagger_z K(z, \zeta) = 0, \qquad \textit{for} \qquad \zeta \neq z. \tag{2.10}$$

Theorem 2.5 (Cauchy's integral formula). *Let G be a domain in* C^2 *and let* $\varphi \in C^\infty(\overline{G}, S^+)$*. Then*

$$\varphi(z) = -\frac{1}{2\pi^2}\int_G K^\dagger(z,\zeta)D\varphi(\zeta)dV(\zeta) + \frac{1}{2\pi^2}\int_{\partial G} K^\dagger(z,\zeta)(\gamma\varphi)(\zeta)d\sigma(\zeta), \quad z \in G, \tag{2.11}$$

where $\gamma = \gamma_{\partial G}|S^+$ *and* $d\sigma$ *is the surface measure on* ∂G*.*

These are proved, for example, in Proposition 2.1 and Theorem 2.2 of [8].

2.3. Local solutions

Theorem 2.6. *Given an odd spinor with compact support* $\psi \in C_c^\infty(\mathrm{C}^2, S^-)$*, there is a solution* $\phi \in C^\infty(\mathrm{C}^2, S^+)$ *of the equation*

$$D\phi(z) = \psi(z), \qquad z \in \mathrm{C}^2. \tag{2.12}$$

Proof. It is proved in [8] that

$$\phi(z) = \frac{1}{2\pi^2}\int_{\mathrm{C}^2} K^\dagger(z,\zeta)\psi(\zeta)dV(\zeta) \tag{2.13}$$

solves the equation $D\phi = \psi$. □

2.4. Eigen spinors of the tangential Dirac operator

The Dirac operator D has the polar decomposition

$$D = \gamma_+\left(\frac{\partial}{\partial n} - \partial\!\!\!/\right). \tag{2.14}$$

The eigenvalues of the tangential Dirac operator $\partial\!\!\!/$ on $|z| = 1$ are

$$\frac{n}{2}, \quad -\frac{n+3}{2}; \qquad n = 0, 1, \ldots,$$

and the multiplicity of each eigenvalue is equal to $(n+1)(n+2)$.

In [9] we gave a complete orthonormal system of eigenspinors $\{\phi^{\pm(n,m,l)}\}$ of $\partial\!\!\!/$ in $L^2(\{|z| = 1\}, S^+)$:

$$\begin{aligned} \partial\!\!\!/\phi^{(n,m,l)} &= \frac{n}{2}\phi^{(n,m,l)} \\ \partial\!\!\!/\phi^{-(n,m,l)} &= -\frac{n+3}{2}\phi^{-(n,m,l)}, \end{aligned} \tag{2.15}$$

for $l = 0, 1, \ldots, n+1$, $m = 0, 1, \ldots, n$, $n = 0, 1, \ldots$.

$\{\phi^{\pm(n,m,l)}\}$ are extended to $\mathrm{C}^2 \setminus \{0\}$ by the homogeneity relations

$$\phi^{(n,m,l)}(z) = |z|^n\phi^{(n,m,l)}\left(\frac{z}{|z|}\right), \tag{2.16}$$

$$\phi^{-(n,m,l)}(z) = |z|^{-(n+3)}\phi^{-(n,m,l)}\left(\frac{z}{|z|}\right), \tag{2.17}$$

for $z \neq 0$. Then,

$$\begin{aligned} D\phi^{(n,m,l)}(z) &= 0, \qquad \text{on } \mathrm{C}^2. \\ D\phi^{-(n,m,l)}(z) &= 0, \qquad \text{on } \mathrm{C}^2 \setminus \{0\}. \end{aligned} \tag{2.18}$$

2.5. Effects of conformal transformations

Here we look at the effect of conformal transformations on the system $\{\phi^{\pm(n,m,l)}\}$. Let $f : U \longrightarrow \mathrm{R}^4$ be a conformal transformation. f induces a $Spin(4)$-equivariant map $f_\flat$ of $Spin(4)$-principal bundles and it yields a bundle isometry $f' = \Delta(f_\flat) : S \longrightarrow S'$ of the associated spinor bundles. The Dirac operator is conformally covariant, that is, if D' is the Dirac operator corresponding to the metric g'; $f^*g' = e^{2u}g$, then

$$D'_{f(z)} = F \cdot D_z \cdot F^{-1}, \tag{2.19}$$

where u is a smooth function on U and $F = e^{-\frac{3}{2}u} f'$, [6 , 10].

Now let U be a domain containing the disk $\{|z| \le 1\}$. Then we can verify

$$\partial\!\!\!/'_{f(z)} = \pm F\, \partial\!\!\!/_z\, F^{-1} = \pm f'\, \partial\!\!\!/_z\, (f')^{-1}, \qquad \text{on } |z| = 1. \tag{2.20}$$

Hence on the sphere $f(\{|z| = 1\}) \subset f(U)$ the eigenvalues of $\partial\!\!\!/$ are $-\frac{n+3}{2}$, $\frac{n}{2}$, $n = 0, \pm 1, \ldots$, if f is orientation preserving, while they change to $\frac{n+3}{2}$, $-\frac{n}{2}$, $n = 0, \pm 1, \ldots$, if f is orientation reversing. The corresponding eigenspinors become $f'\phi^{\pm(n,m,l)}$, that were extended to $\mathrm{R}^4 \setminus f(0)$ by $F\phi^{\pm(n,m,l)}$. In particular, by a coordinate change $T \in SO(4)$ we have the same eigenvalues of $\partial\!\!\!/$ and the eigenspinors are $\phi^{\pm(n,m,l)} \circ T$, and our theory is independent of the choice of the complex structure $\mathrm{C}^2 \simeq \mathrm{R}^4$. By the transformation $f(z) = c + rz$, $r > 0$, we find that the eigenspinors on $|z - c| = r$ are given by

$$r^{\mp(n+\frac{3}{2})}\phi^{\pm(n,m,l)}(z - c).$$

On the other hand by the inversion $f(z) = -\frac{\overline{z}}{|z|}$, we have

$$F\phi^{\pm(n,m,l)}(z) = \overline{|z|^3\gamma(z) \cdot \phi^{\pm(n,m,l)}(z)}. \tag{2.21}$$

Note that $\overline{|z|^3\gamma(z)\cdot\phi^{(n,m,l)}}$ belongs to the eigenvalue $-\frac{n}{2}$.

Having verified the covariance of our theory on R^4 under conformal transformations we can extend it to a manifold which is locally R^4 and patched together by conformal transformations, that is, to a conformally flat 4-manifold. For example, S^4 is obtained by patching up two copies of C^2 together by the inversion $w = f(z) = \frac{\overline{z}}{|z|^2}$. We shall denote these two local coordinates by C^2_z and C^2_w. f has the conformal weight $u = -\log|z|^2$; $f^*(dwd\overline{w}) = \frac{1}{|z|^4}(dzd\overline{z})$. Therefore an even spinor ϕ on a subset U of S^4 is a pair of $\phi \in C^\infty(U \cap \mathrm{C}^2_z \times \Delta^+)$ and $\widehat{\phi} \in C^\infty(U \cap \mathrm{C}^2_w \times \Delta^-)$ such that

$$\widehat{\phi}(w) = (f'\phi)(f(z)) = \overline{|z|^3(\gamma_+ \cdot \phi)(z)}, \tag{2.22}$$

for $w = f(z)$.

The Cauchy kernel on C^2_w has the form

$$\widehat{K}^\dagger(w, \eta) = -\overline{|z|^3\gamma_+(z)K^\dagger(z,\zeta)\gamma_+(\zeta)}, \qquad w = f(z),\, \eta = f(\zeta). \tag{2.23}$$

2.6. Expansion of the Cauchy kernel

We proved in [8] that the Cauchy kernel has the following expansion by the spinors $\phi^{\pm(n,m,l)}(z-c)$.

Theorem 2.7. 1. *For* $|z-c| < |\zeta - c|$,

$$K^{\dagger}(z,\zeta)\cdot\gamma_{+}(\zeta-c) = \sum_{n}\sum_{l=0}^{n+1}\sum_{m=0}^{n} |\zeta-c|^{-(2n+3)}\overline{\phi^{(n,m,l)}(\zeta-c)}\otimes\phi^{(n,m,l)}(z-c).$$

The convergence is uniform on any compact subset of $\{|z-c|<|\zeta-c|\}$.

2. *For* $|z-c| > |\zeta-c|$,

$$K^{\dagger}(z,\zeta)\cdot\gamma_{+}(\zeta-c) = -\sum_{n}\sum_{l=0}^{n+1}\sum_{m=0}^{n} |\zeta-c|^{2n+3}\overline{\phi^{-(n,m,l)}(\zeta-c)}\otimes\phi^{-(n,m,l)}(z-c).$$

The convergence is uniform on any compact subset of $\{|z-c|>|\zeta-c|\}$.

2.7. Meromorphic spinors

The Cauchy integral formula and the expansion of the Cauchy kernel in 2.2 and 2.6 yield the Laurent expansion of a harmonic spinor [2, 8].

Theorem 2.8. *Let* φ *be a smooth even spinor on the annular region* $0 \le r < |z-c| < R \le \infty$ *such that* $D\varphi = 0$. *Then we have the expansion*

$$\varphi(z) = \sum_{(n,m,l)} C_{(n,m,l)}\,\phi^{(n,m,l)}(z-c) + \sum_{(n,m,l)} C_{-(n,m,l)}\,\phi^{-(n,m,l)}(z-c), \tag{2.24}$$

for $r < |z-c| < R$. *The coefficients are uniquely determined by* (n,m,l) *and* c *and are given by*

$$C_{\pm(n,m,l)} = \frac{\rho^{\mp(2n+3)}}{2\pi^2}\int_{|\zeta-c|=\rho} <\varphi(\zeta)\,,\,\phi^{\pm(n,m,l)}(\zeta-c)> \sigma(d\zeta) \tag{2.25}$$

for any ρ *such that* $r<\rho<R$.

In the expansion of φ in (2.26) the second part

$$\sum_{(n,m,l)} C_{-(n,m,l)}\,\phi^{-(n,m,l)}(z-c) \tag{2.26}$$

is called *principal part* of φ at c.

Definition 2.9. Let G be a domain in C^2 and let E be a discrete subset of G. A harmonic spinor φ on $G\setminus E$ is said to be *meromorphic* on G if its principal part has only finitely many terms at every point of E. A point of E is called a pole of φ.

Let φ be a meromorphic spinor with a pole at $p \in G$. Let f be a conformal transformation. $f'\varphi$ is expanded around $f(p)$ by a series of $f'\phi^{\pm(n,m,l)}$, and the coefficients are given by the corresponding coefficients $C_{\pm(n,m,l)}$ of the expansion of φ around p. Thus the order of a pole of a meromorphic spinor is invariant by conformal transformations.

S^4 is obtained by patching up C^2_z and C^2_w together by the inversion $w = f(z) = \frac{\overline{z}}{|z|^2}$. On C^2_w the basis vectors of the Laurent expansion at infinity $\widehat{0}$ are

$$\widehat{\phi}^{\pm(m,l,k)} = \overline{|z|^3(\gamma_+ \cdot \phi^{\pm(m,l,k)})(z)},$$

with $\widehat{\phi}^{-(m,l,k)} \in O(|w|^m)$ and $\widehat{\phi}^{(m,l,k)} \in O(|w|^{-(m+3)})$. So, on a neighborhood U of $\widehat{0}$, $\varphi \in \mathcal{N}(U)$ has the Taylor expansion

$$\varphi(w) = \sum C_{-(m,l,k)} \widehat{\phi}^{-(m,l,k)}(w). \tag{2.27}$$

3. Runge's approximation theorem

3.1. Approximation of harmonic spinors on a compact set

First we note the following;

1. For a S^--valued Radon measure μ on $G \subset \mathrm{C}^2$, put
$$K^\dagger\mu(z) = \int K^\dagger(z,\zeta)\mu(d\zeta). \tag{3.1}$$
For $\mu(d\zeta) = \phi(\zeta)\, dV(\zeta)$ with $\phi \in \Gamma(S^-)$, we shall abbreviate to $K^\dagger\phi(z)$. Then
$$K^\dagger\mu(z) \in C^\infty(\mathrm{C}^2 \setminus G, S^+), \qquad DK^\dagger\mu(z) = 0.$$
2. For a ${}^t\overline{S^+}$-valued Radon measure ν on $G \subset \mathrm{C}^2$, put
$$\nu K^\dagger(\zeta) = \int \nu(dz)K^\dagger(z,\zeta). \tag{3.2}$$
$\nu K^\dagger(\zeta)$ belongs to $C^\infty(\mathrm{C}^2 \setminus G, {}^t\overline{S}^-)$, and $D^\dagger\, {}^t\overline{(\nu K^\dagger)}(\zeta) = 0$.
3. $$\int \nu K^\dagger(\zeta)\mu(d\zeta) = \int \nu(dz)K^\dagger\mu(z) \tag{3.3}$$
if $Supp[\mu] \cap Supp[\nu] = \phi$.

Theorem 3.1. *Let G be a domain in C^2 and K be a compact subset of G. Then any harmonic spinor defined in a neighborhood of K is approximated uniformly on K by harmonic spinors on G if and only if the open set $G \setminus K$ has no component which is relatively compact in G.*

Proof. Sufficiency: We shall show that every ${}^t\overline{S^+}$-valued Radon measure ν on K which annihilates the harmonic (even) spinors on G annihilates also the harmonic (even) spinors in a neighborhood of K, (then use Hahn-Banach).

Let ν be a ${}^t\overline{S^+}$ valued Radon measure on K that annihilates the harmonic (even) spinors on G.

(1) Put

$$g(\zeta) = \nu K^{\dagger}(\zeta) = \int \nu(dz) K^{\dagger}(z,\zeta), \tag{3.4}$$

and $f(\zeta) = {}^t\overline{g}(\zeta)$.

f is well defined and $D^{\dagger} f(\zeta) = 0$ for $\zeta \in \mathrm{C}^2 \setminus K$. Since $D_z K^{\dagger}(z,\zeta) = 0$ for $\zeta \neq z$, $f(\zeta) = 0$ on $\mathrm{C}^2 \setminus G$ from the assumption. Hence, by the unique continuation property of (odd) harmonic spinors, $f(\zeta) = 0$ in every component of $\mathrm{C}^2 \setminus K$ which intersects $\mathrm{C}^2 \setminus G$.

Next, since $\phi^{\lambda}(z)$, $\lambda = (n, m, l)$, is a harmonic spinor on C^2, we have

$$\int \nu(dz)\phi^{\lambda}(z) = 0.$$

So Theorem 2.7 implies that

$$g(\zeta)\gamma_+(\zeta) = \int \nu(dz) K^{\dagger}(z,\zeta)\gamma_+(\zeta) = 0,$$

for $|\zeta| > \sup_{z \in K} |z|$. Hence $f(\zeta) = 0$ in the unbounded component of $\mathrm{C}^2 \setminus K$. Since $G \setminus K$ has no component which is relatively compact in G we conclude that $f = 0$ and $g = 0$ on $\mathrm{C}^2 \setminus K$.

(2) Let φ be a harmonic spinor on a neighborhood ω of K, and choose a smooth function u with compact support such that $u = 1$ on K.

Since $u\varphi$ has compact support, we have, from integral formula (2.11),

$$\begin{aligned} \varphi(z) = u(z)\varphi(z) &= -\frac{1}{2\pi^2} K^{\dagger} D(u\varphi)(z) \\ &= -\frac{1}{2\pi^2} K^{\dagger}(\, du \cdot \varphi)(z), \quad z \in \omega. \end{aligned}$$

By (3.3),

$$\begin{aligned} \int \nu(dz)\varphi(z) &= -\int \nu(dz) \frac{1}{2\pi^2} K^{\dagger}(\, du \cdot \varphi)(z) \\ &= -\frac{1}{2\pi^2} \int \nu K^{\dagger}(\zeta)(\, du \cdot \varphi)(\zeta) dV(\zeta). \end{aligned}$$

Since $g = \nu K^{\dagger} = 0$ on $\mathrm{C}^2 \setminus K$ and the support of $(du \cdot \varphi)(\zeta) dV(\zeta)$ is contained in $\mathrm{C}^2 \setminus K$, the last integral vanishes. Thus ν annihilates the harmonic spinors on K.

Necessity: We assume that $G \setminus K$ has a component H such that $\overline{H}$ is compact in G. Then the boundary of H is a subset of K and the maximum principle for subharmonic functions yields

$$\sup_{\overline{H}} |\phi| \leq \sup_{K} |\phi|, \qquad \text{for every harmonic spinor } \phi \text{ on } G, \tag{3.5}$$

where $|\phi| = (< \phi, \phi >)^{\frac{1}{2}}$. Let φ be a harmonic spinor defined in a neighborhood of K. By the assumption we can choose a sequence of harmonic spinors φ_n on G so that $\varphi_n \longrightarrow \varphi$ uniformly on K. (3.5) applied to $\varphi_n - \varphi_m$ implies that φ_n converges uniformly on $\overline{H}$ to a limit Φ. Then $\Phi = \varphi$ on the boundary of H, and

Φ is a harmonic spinor in H and is continuous in $\overline{H}$. In particular, we can choose $\varphi(z) = K^{\dagger}(z, \zeta)$ for a $\zeta \in H$. Then $\Phi(z) = K^{\dagger}(z, \zeta)$ on the boundary of H. Hence $K^{\dagger}(\quad, \zeta)$ which is a harmonic spinor in $H \setminus \zeta$ is extended to a harmonic spinor in H. This is a contradiction. $\square$

3.2. Global solutions of $D\phi = \psi$

Theorem 3.2. *Let G be an open set in C^2. Given a $\psi \in C^{\infty}(G, S^{-})$, there is a solution $\phi \in C^{\infty}(G, S^{+})$ of the equation*

$$D\phi(z) = \psi(z), \qquad z \in G. \tag{3.6}$$

Proof. Choose an increasing sequence of compact sets $K_j \subset G$ such that every compact subset of G is contained in some K_j. We may suppose that no component of $G \setminus K_j$ is relatively compact in G. If not take the union of K_j and all components of $G \setminus K_j$ that are relatively compact in G. Let h_j be a smooth function with compact support in G such that $h_j = 1$ in a neighborhood of K_j. Put $f_1 = h_1$, $f_j = h_j - h_{j-1}$. Then f_j has compact support and $f_j = 0$ in a neighborhood of K_{j-1} and $\sum f_j = 1$. From the local existence theorem there exists a $\phi_j \in C^{\infty}(\mathrm{C}^2, S^{+})$ such that

$$D\phi_j = f_j \psi.$$

This means in particular that ϕ_j is a harmonic spinor in a neighborhood of K_{j-1}. By Theorem 3.1 we can find a harmonic spinor φ_j on G so that $\sup_{K_{j-1}} |\varphi_j - \phi_j| < 2^{-j}$. Then the sum

$$\phi = \sum (\varphi_j - \phi_j)$$

is uniformly convergent on every compact subset of G. For each k the sum from $k+1$ to ∞ converges uniformly on K_k to a harmonic spinor in the interior of K_k. Hence $\phi \in C^{\infty}(G, S^{+})$ and we have

$$D\phi = \sum D\phi_j = \sum f_j \psi = \psi. \qquad \square$$

4. Cohomology Groups of Harmonic Spinors

4.1. Cohomology on conformally flat manifolds

Let (M, g) be a riemannian 4-manifold with conformally flat metric. We suppose that M has a spin structure and we fix it.

There exists a system of coordinate neighborhoods $(U_\lambda, \chi_\lambda)$ in M such that each $G_\lambda = \chi_\lambda(U_\lambda)$ is a domain in C^2 with local coordinate $\{z_1^\lambda, z_2^\lambda\}$ and such that the transition function $f_{\mu\lambda} = \chi_\mu \chi_\lambda^{-1}$ is a conformal transformation:

$$f_{\mu\lambda}^{*}\left(dz_1^\mu d\overline{z}_1^\mu + dz_2^\mu d\overline{z}_2^\mu\right) = e^{2u_{\mu\lambda}}\left(dz_1^\lambda d\overline{z}_1^\lambda + dz_2^\lambda d\overline{z}_2^\lambda\right), \tag{4.1}$$

where $u_{\mu\lambda}$ is a smooth function on $G_\mu \cap G_\lambda$. We have also

$$(\chi_\lambda^{-1})^{*} g = e^{2u_\lambda}(dz_1^\lambda d\overline{z}_1^\lambda + dz_2^\lambda d\overline{z}_2^\lambda), \tag{4.2}$$

with u_λ a smooth function on G_λ, and $u_{\mu\lambda} = u_\lambda - u_\mu$ on $G_\lambda \cap G_\mu$.

The Dirac operator is conformally covariant. Let D be the Dirac operator on (M, g) and let D_λ be the Dirac operator on $G_\lambda \subset \mathrm{C}^2$ that we have discussed hitherto, then

$$D \cdot F_\lambda = F_\lambda \cdot D_\lambda, \tag{4.3}$$

where $F_\lambda = e^{-\frac{3}{2}u_\lambda}(\chi_\lambda)'$, see 2.5.

On each open set U contained in a coordinate neighborhood the Dirac operator

$$D : C^\infty(U, S^+) \longrightarrow C^\infty(U, S^-) \tag{4.4}$$

is surjective from Theorem 3.2. Thus every open covering $\mathcal{U}$ by open subsets contained in coordinate neighborhoods is a Leray covering and we have

$$H^k(M, \mathcal{N}) = H^k(\mathcal{U}, \mathcal{N}) \tag{4.5}$$

for every $k \geq 0$.

The structure of a Fréchet space on $C^\infty(G, S^\pm)$ is defined by the uniform convergence of all the derivatives on compact subsets that are contained in coordinate neighborhoods of G. On the vector subspace of harmonic spinors $\mathcal{N}(G)$ the induced topology coincides with the topology of uniform convergence on compact subsets. We endow $H^1(M, \mathcal{N}) = H^1(\mathcal{U}, \mathcal{N})$ with the structure of a Fréchet space in an obvious way.

From (1.2) we know that $H^k(M, \mathcal{N}) = 0$ for $k \geq 2$.

Theorem 4.1.

$$H^1(G, \mathcal{N}) = 0, \tag{4.6}$$

for any open subset G in C^2.

The assertion folows from Theorem 3.2.

Theorem 4.2.

$$H^1(S^4, \mathcal{N}) = 0. \tag{4.7}$$

Proof. Let $U_0 = S^4 \setminus \infty = \mathrm{C}^2_z$ and $U_1 = S^4 \setminus 0$. U_1 is conformally equivalent to C^2_w by the Kelvin inversion $w = \frac{\overline{z}}{|z|^2}$. It follows from Theorem 4.1 that $\{U_0, U_1\}$ is a Leray covering of S^4. Let $f_{01} \in \mathcal{N}(U_0 \cap U_1)$. The Laurent expansion of f_{01} on $U_0 \cap U_1 = \mathrm{C}^2 \setminus 0$ becomes

$$f_{01}(z) = \sum_{m,l,k} C_{(m,l,k)}\, \phi^{(m,l,k)}(z) + \sum_{m,l,k} C_{-(m,l,k)}\, \phi^{-(m,l,k)}(z), \qquad z \neq 0.$$

Put

$$f_0(z) = \sum_{m,l,k} C_{(m,l,k)}\, \phi^{(m,l,k)}(z), \quad \text{and} \quad f_1(z) = \sum_{m,l,k} C_{-(m,l,k)}\, \phi^{-(m,l,k)}(z).$$

Then $f_0 \in \mathcal{N}(U_0)$, while f_1 viewed on the coordinate neighborhood C^2_w, $w = \frac{\overline{z}}{|z|^2}$, is harmonic, see the discussion at the end of 2.7. Hence $f_1 \in \mathcal{N}(U_1)$, and $f_{01} = f_0 - f_1$. Therefore $H^1(S^4, \mathcal{N}) = 0$. □

We would like to pose the following conjecture:

$$H^1(M,\mathcal{N}) = 0,$$

for every non-compact conformally flat spin manifold M.

Theorem 4.3. *Let G' be an open subset of M and G be a relatively compact open subset of G'. Then the restriction map*

$$r^* : H^1(G',\mathcal{N}) \longrightarrow H^1(G,\mathcal{N})$$

has a finite dimensional image.

Proof. This follows from Theorem 2.3 and the Schwartz lemma. The latter says that the image of a homomorphism of Fréchet spaces which is the sum of an epimorphism and a compact morphism has a finite codimension. Let $\mathcal{U}$ be an open covering of G'. Let V be the image of the map

$$r \oplus \delta : \ Z^1(\mathcal{U},\mathcal{N}) \oplus C^0(\mathcal{U} \cap G,\mathcal{N}) \longrightarrow Z^1(\mathcal{U} \cap G,\mathcal{N}),$$

where δ is the coboundary map and r is the restriction map. Then by the Schwartz lemma,

$$\frac{V}{\mathrm{Im}(r \oplus \delta - r \oplus 0)} = \frac{V}{\delta C^0(\mathcal{U} \cap G,\mathcal{N})} = \mathrm{Im}\, r^*$$

is finite dimensional. $\square$

In particular if M is compact the theorem yields the finiteness of $\dim H^1(M,\mathcal{N})$. But this is obvious from the finiteness of $\dim \mathrm{coker} D$.

4.2. Existence of a non-trivial meromorphic spinor

Let M be a conformally flat 4-manifold and $(U_\lambda, \chi_\lambda)$ be a system of coordinate neighborhoods such that $G_\lambda = \chi_\lambda(U_\lambda)$ is a domain in C^2. A smooth spinor φ on $M \setminus E$, E being a discrete subset, is called a *meromorphic spinor* on M with poles at E if, for each λ, $\varphi_\lambda = (\chi_\lambda)'\varphi$ is a meromorphic spinor on $G_\lambda \subset \mathrm{C}^2$ with poles at $G_\lambda \cap \chi_\lambda(E)$. This is equivalent to saying that a family of meromorphic spinors φ_λ on $G_\lambda \subset \mathrm{C}^2$ such that $\varphi_\mu = (\chi_{\mu\lambda})'\varphi_\lambda$ defines a meromorphic spinor on M.

Theorem 4.4. *Let G be a relatively compact open subset of M. For every point $p \in G$ there is a meromorphic spinor on G which has a pole at p and is smooth on $G \setminus p$.*

Proof. Let (U_1, χ_1) be a coordinate neighborhood of p such that $\chi_1(p) = 0$ and $U_2 = M \setminus \{p\}$. Let

$$d = \dim\, \mathrm{Im}(H^1(M,\mathcal{N}) \longrightarrow H^1(G,\mathcal{N})).$$

We take a set Λ constituting $d+1$ indices from the set of indices (m,l,k); $m \geq 1$, $0 \leq l \leq m$, $0 \leq k \leq m+1$, and we consider spinors $\{\phi^{-(n,m,l)}; (n,m,l) \in \Lambda\}$.

These are meromorphic spinors on U_1 and define a cocycle on $U_1 \cap U_2 = U_1 \setminus \{p\}$. Restricted on G, the cocycles are linearly dependent, hence there is a linear combination

$$\sum_{(n,m,l)\in\Lambda} A_{(n,m,l)}\phi^{-(n,m,l)}$$

that belongs to the coboundary, that is, there exist $\xi_i \in \mathcal{N}(U_i)$; $i = 1.2$, such that

$$\sum_{(n,m,l)\in\Lambda} A_{(n,m,l)}\phi^{-(n,m,l)} = \xi_2 - \xi_1.$$

Therefore a spinor φ which coincides on $U_1 \cap G$ with $\sum_{(n,m,l)\in\Lambda} A_{(n,m,l)}\phi^{-(n,m,l)} + \xi_1$, and which is equal to ξ_2 on $U_2 \cap G$, gives a meromorphic spinor on G with the only pole at p. □

Let $\mathcal{U} = \cup_{i\in I} U_i$ be an open covering of M. A family $(\varphi_i)_{i\in I}$ of meromorphic spinors φ_i on U_i is called a *Mittag-Leffler distribution* on M if the differences $\varphi_{ij} = \varphi_i - \varphi_j$ are harmonic spinors on $U_i \cap U_j$.

The family of differences φ_{ij} defines a cocycle and this cocycle is a coboundary precisely when there exists a meromorphic spinor φ on M such that for each $i \in I$ the difference $\varphi - \varphi_i$ is harmonic on U_i.

From Theorems 4.1 and 4.2 we have the following Mittag-Leffler type theorems.

Theorem 4.5. *Let G be an open subset of* C^2. *Let $(\varphi_i)_{i\in I}$ be a Mittag-Leffler distribution on G. Then there exists a meromorphic spinor φ on G such that for each $i \in I$ the difference $\varphi - \varphi_i$ is harmonic on U_i.*

Theorem 4.6. *Let $(\varphi_i)_{i\in I}$ be a Mittag-Leffler distribution on S^4. Then there exists a meromorphic spinor φ on S^4 such that for each $i \in I$ the difference $\varphi - \varphi_i$ is harmonic on U_i.*

4.3. Serre duality

For an open subset U, $\mathcal{D}'(U, S^{\pm})$ denotes the set of $S^{\pm}$-valued distributions on U, and $\mathcal{D}'(S^{\pm})$ denotes the sheaf of $S^{\pm}$-valued distributions. For $T \in \mathcal{D}'(U, S^{\pm})$ and $\varphi \in C_c^\infty(U, \mathcal{S}^{\mp})$, respectively, we have, by the definition,

$$DT[\varphi] = -T[D^\dagger \varphi], \quad D^\dagger T[\varphi] = -T[D\varphi],$$

respectively. $\mathcal{D}'(M, S^{\pm})$ is a Fréchet space which is dual to $C_c^\infty(M, \mathcal{S}^{\pm})$, and $D^\dagger$ is the transposition of D.

We have the following exact sequence:

$$0 \longrightarrow \mathcal{N}^\dagger \longrightarrow \mathcal{D}'(S^-) \xrightarrow{D^\dagger} \mathcal{D}'(S^+) \longrightarrow 0. \tag{4.8}$$

Here we used Weyl's lemma to have the kernel $\mathcal{N}^\dagger$.

Let $\mathcal{E}'(M, S^{\pm}) = \Gamma_c(M, \mathcal{D}'(S^{\pm}))$ be the space of $S^{\pm}$-valued distributions with compact supports and let $H_c^k(M, \mathcal{N}^\dagger)$, $k = 0, 1$, be the cohomology groups with compact supports. $\mathcal{E}'(M, S^{\pm})$ is the dual of the Fréchet space $C^\infty(M, S^{\pm})$. From the short exact sequences (1.1) and (4.10) we have the following exact sequences of cohomology groups:

$$0 \longrightarrow H^0(M,\mathcal{N}) \quad \longrightarrow C^\infty(M,\mathcal{S}^+) \xrightarrow{D} \quad C^\infty(M,\mathcal{S}^-) \longrightarrow H^1(M,\mathcal{N}),$$

$$0 \longleftarrow H^1_c(M,\mathcal{N}^\dagger) \quad \longleftarrow \mathcal{E}'(M,S^+) \xleftarrow{D^\dagger} \quad \mathcal{E}'(M,S^-) \longleftarrow H^0_c(M,\mathcal{N}^\dagger) = 0.$$

$H^0_c(M,\mathcal{N}^\dagger) = 0$ from Theorem 2.2. Suppose that $H^1(M,\mathcal{N}) = 0$. Then D, being surjective, becomes a homomorphism of Fréchet spaces by the Banach theorem.

Theorem 4.7. *Suppose $H^1(M,\mathcal{N}) = 0$. Then the dual of $H^0(M,\mathcal{N})$ is isomorphic to $H^1_c(M,\mathcal{N}^\dagger)$.*

5. Divisors of Meromorphic Spinors

5.1. Divisors and meromorphic spinors

We shall consider the cohomology groups of meromorphic spinors on a compact conformally flat spin manifold M, and prove an analogy of the Riemann-Roch theorem. First we note that there is a meromorphic spinor with a pole at a prescribed point on M, this was proved in Theorem 4.4.

A divisor on a space X is a mapping $E : X \longrightarrow \mathbf{Z}$ such that for any compact subset K there are finitely many points $c \in K$ with $E(c) \neq 0$. With respect to addition the set of all divisors forms an abelian group $Div(X)$. There is a partial ordering on $Div(X)$; for $E, E' \in Div(X)$, set $E \leq E'$ if $E(c) \leq E'(c)$ for every $c \in X$.

Since M is compact, for any $E \in Div(M)$, there are only finitely many $x \in M$ such that $E(x) \neq 0$. Then we define the degree

$$\deg :\ Div(M) \longrightarrow \mathrm{Z},$$

by $\deg E = \sum_{x\in M} E(x)$.

We shall now define an ordered set of indices to enumerate the basis of spinors of the Laurent expansion: $\phi^{\pm(n,m,l)}$. We introduce a triplet $\lambda = (n,m,l)$, $0 \leq m \leq n$, $0 \leq l \leq n+1$. The lexicographic order for two triplets λ is defined by $\lambda \geq \lambda'$ if either (i)$n > n'$, or (ii)$n = n'$, $m > m'$, or (iii) $n = n'$, $m = m'$ and $l \geq l'$. We introduce also the notation $-\lambda = -(n,m,l)$, and define $-\lambda \geq -\lambda'$ if $\lambda \leq \lambda'$. The smallest positive is $o_+ = (0,0,0)$ and the largest negative is $o_- = -(0,0,0)$.

We denote by $\mathcal{Z}$ the set of all triplets λ and $\mathcal{Z}_{\geq 0_+}$ (resp. $\mathcal{Z}_{\leq 0_-}$) the set of all $\lambda \geq o_+$ (resp. $-\lambda \leq o_-$). We denote $|\pm\lambda| = \pm n$ for $\pm\lambda = \pm(n,m,l)$, and put $\mathcal{Z}_{\pm n} = \{\pm\lambda = \pm(n,m,l);\, 0 \leq m \leq n, 0 \leq l \leq n+1\}$. Note that, by convention, $+0 \neq -0$.

For a meromorphic spinor φ on an open set $G \subset M$ and $c \in G$, the Laurent expansion at c becomes

$$\varphi(z) = \sum_{\lambda \in \mathcal{Z}_{\geq 0}} C_\lambda \, \phi^\lambda(z-c) + \sum_{-\lambda \in \mathcal{Z}_{\leq 0}} C_{-\lambda} \, \phi^{-\lambda}(z-c), \qquad r < |z-c| < R. \quad (5.1)$$

We define

$$\mathrm{ord}_c(\varphi) := \begin{cases} 0, & \text{if } C_{-\lambda} = 0 \text{ for all } -\lambda \leq 0_- \text{ and } C_\lambda \neq 0 \text{ for some } \lambda \in \mathcal{Z}_0 \\ k, & \text{if } C_{-\lambda} = 0 \text{ for all } -\lambda \leq 0_- \text{ , } C_\lambda = 0 \text{ for all } |\lambda| \leq k-1, \\ & \quad \text{and } C_\lambda \neq 0 \text{ for some } \lambda \in \mathcal{Z}_k \\ -(k+1), & \text{if } C_{-\lambda} = 0 \text{ for } |-\lambda| \leq -(k+1), \\ & \quad \text{and } C_{-\lambda} \neq 0 \text{ for some } -\lambda \in \mathcal{Z}_{-k} \\ \infty, & \text{if } \varphi \equiv 0 \text{ in a neighborhod of } c \end{cases}$$

The divisor of a meromorphic spinor φ not identically 0 is defined by

$$(\varphi) = \sum_{c \in G} ord_c(\varphi) \cdot c. \quad (5.2)$$

For example, from (2.22) we know that each $\phi^{\pm(n,m,l)}$ gives a meromorphic spinor on S^4. Let $\widehat{0}$ denote the point at infinity: $\mathrm{C}^2_z \cup \widehat{0} = S^4$. We have

$$\begin{aligned} (\, \phi^{(n,m,l)} \,) &= n \cdot 0 - (n+1) \cdot \widehat{0} \\ (\, \phi^{-(n,m,l)} \,) &= -(n+1) \cdot 0 + n \cdot \widehat{0}. \end{aligned}$$

Let E be a Divisor. For an open set $U \subset M$ we define

$$\mathcal{L}_E(U) = \{\varphi \in \mathcal{M}(U);\, ord_x(\varphi) \geq -E(x) \quad \text{for } \forall x \in U.\}$$

Example. We have for $E = np$, $p \in M$,

$$\begin{aligned} \mathcal{L}_E(U) &= \{\varphi = \sum_{|-\lambda| \geq -(n-1)} C_{-\lambda}\phi^{-\lambda}(z) + \sum_{\lambda \geq 0_+} C_\lambda \phi^\lambda(z) \,\} \\ \mathcal{L}_{-E}(U) &= \{\, \varphi = \sum_{|\lambda| \geq n} C_\lambda \phi^\lambda(z) \,\} \end{aligned}$$

in a local coordinate around p.

Let $p \in M$. The skyscraper sheaf C_p is defined by

$$\mathrm{C}_p(U) = \begin{cases} \mathrm{C}, & \text{if } p \in U, \\ 0, & \text{if } p \notin U, \end{cases}$$

Let $E \in Div(M)$ and $p \in M$. We look at p also as a divisor, $p \in Div(M)$. Then $E' = E + p \in Div(M)$. We define a sheaf homomorphism

$$\rho : \mathcal{L}_{E'} \longrightarrow \mathrm{C}_p$$

as follows. For an open set U, if $p \neq U$ then ρ_U is the zero homomorphism. If $p \in U$ and $\varphi \in \mathcal{L}_{E'}(U)$, then φ admits a Laurent expansion around p,

$$\varphi(z) = \sum_{|-\lambda| \geq -k} C_{-\lambda}\phi^{-\lambda} + \sum_{\lambda \geq 0_+} C_\lambda \phi^\lambda,$$

where $E(p) = k$, hence $E'(p) = k+1$. Set $\rho_U(\varphi) = C_{-\lambda_0}$ for the smallest $-\lambda_0$ such that $C_{-\lambda_0} \neq 0$ and $|-\lambda_0| = -k$. Then ρ is a sheaf homomorphism and we have the short exact sequence

$$0 \longrightarrow \mathcal{L}_E \longrightarrow \mathcal{L}_{E+p} \longrightarrow \mathrm{C}_p \longrightarrow 0. \tag{5.3}$$

Therefore we have the following exact sequence of cohomologies:

$$\begin{aligned} 0 \longrightarrow H^0(M, \mathcal{L}_E) &\longrightarrow H^0(M, \mathcal{L}_{E+p}) \longrightarrow \mathrm{C} \\ &\longrightarrow H^1(M, \mathcal{L}_E) \longrightarrow H^1(M, \mathcal{L}_{E+p}) \longrightarrow 0. \end{aligned} \tag{5.4}$$

Lemma 5.1. *Suppose M is compact and let $E \leq E'$ be divisors. Then we have an epimorphism*

$$H^1(M, \mathcal{L}_E) \longrightarrow H^1(M, \mathcal{L}_{E'}) \longrightarrow 0.$$

Theorem 5.2. *Suppose E is a divisor on a compact conformally flat four dimensional spin manifold M. Then*

$$\dim H^0(M, \mathcal{L}_E) - \dim H^1(M, \mathcal{L}_E) = \deg E, \tag{5.5}$$

$$H^p(M, \mathcal{L}_E) = 0 \ \textit{for}\ p \geq 2. \tag{5.6}$$

Proof. For a divisor $E = 0$, the assertion follows from the Atiyah-Singer index theorem (1.3). In fact, for a conformally flat manifold M, the Weyl tensor vanishes from the definition, and the first Pontryagin number $p_1(M)$ is zero.

Let $E \in Div(M)$ and $E' = E + p$. Let $V = Im(H^0(M, \mathcal{L}_{E'}) \longrightarrow \mathrm{C})$ and $W = \mathrm{C}/V$. Then $\dim V + \dim W = 1 = \deg E' - \deg E$. From the exact sequence (5.4) we have,

$$\begin{aligned} \dim H^0(M, \mathcal{L}_{E'}) &= \dim H^0(M, \mathcal{L}_E) + \dim V, \\ \dim H^1(M, \mathcal{L}_{E'}) &= \dim H^1(M, \mathcal{L}_E) - \dim W. \end{aligned}$$

Hence

$$\begin{aligned} &\dim H^0(M, \mathcal{L}_{E'}) - \dim H^1(M, \mathcal{L}_{E'}) - \deg E' \\ &\quad = \dim H^0(M, \mathcal{L}_E) - \dim H^1(M, \mathcal{L}_E) - \deg E. \end{aligned}$$

Thus (5.5) holds for E (resp. E') if it holds for E' (resp. E). In particular (5.5) is true for every divisor $E' \geq 0$. Any divisor may be written as

$$E = p_1 + \ldots + p_m - p_{m+1} - \ldots - p_n.$$

Hence the first assertion is proved by induction. As for the second part, we know $H^p(M, \mathcal{N}) = 0$ for $p \geq 2$, that is, $H^p(M, \mathcal{L}_0) = 0$ for $p \geq 2$. Then, by the same argument using the exact sequence (5.4) as in the first part, we can prove

$$H^p(M, \mathcal{L}_E) = 0 \text{ for } p \geq 2,$$

for every divisor E. □

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Tosiaki Kori
Department of Mathematics
School of Science and Engineering
Waseda University
3-4-1 Okubo, Shinjuku-ku
Tokyo
Japan
e-mail: `kori@waseda.jp`