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ABSTRACT. We discuss the index of the Euclidean Dirac operator D on the unit four–ball subject to a boundary condition on S^3 which is induced by a vector potential A in a canonical way. We show that this index is equal to the index of the Dirac operator D_A coupled with A over the closed four–sphere.

Introduction

The chiral anomaly may be expressed in the following way. Consider a massless Dirac fermion field ψ in S^4, interacting with an external Yang–Mills field A^a_μ. In the second quantized theory the chiral current $j^\mu_5 = \overline{\psi}\gamma^\mu\gamma_5\psi$ is not conserved and the chiral anomaly appears:

$$\partial_\mu j^\mu_5 = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma} tr F_{\mu\nu}F_{\rho\sigma}.$$

It has a local divergence form:

$$\partial_\mu j^\mu_5 = \frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu tr(A_\nu\partial_\rho A_\sigma + \frac{2}{3}A_\nu A_\rho A_\sigma)\,.$$

A well–known formula yields

$$n_+ - n_- = -\frac{1}{32\pi^2}\int d^4x\epsilon^{\mu\nu\rho\sigma} tr F_{\mu\nu}F_{\rho\sigma}\,,$$

where $n_\pm$ is the dimension of the space of right– (resp. left–) handed zero mode spinors and $n_+ - n_-$ is nothing but the index of the half Dirac operator $D_A = D + A$. By the divergence formula this is reduced to the third Chern number of the bundle, which is equal to the instanton number of the connected component of the configuration space $\mathcal{A}/\mathcal{G}$ to which A belongs.

On the other hand, when we consider a field ψ on the unit ball in $\mathbf{R}^4$ we must also take the surface effects into consideration, and the notion of the index depends on the boundary condition which is imposed on the equator. Various classes of boundary conditions were investigated by many authors and it seems nowadays

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that generalized Atiyah–Patodi–Singer boundary conditions are most adequate [2, 3].

The generalized APS boundary conditions form an infinite–dimensional Grassmannian manifold $Gr(S^3)$ of certain spaces of square–integrable spinors on the equator S^3 (see e.g. [8]). The space $Gr(S^3)$ is a geometrical object that is determined by the Dirac operator D, hence by the Riemannian metric on S^4. J. Mickelsson [5] showed that the gauge transformation group $\mathcal{G}$ acts on the Grassmannian. We discussed the relation between the gauge transformation group, the space of connections (i.e., of vector potentials) and the Grassmannian in [4]. We showed that to each vector potential A corresponds an element W_A of $Gr(S^3)$, defined as the image of the base point $H_-^N \in Gr(S^3)$ under the map giving the homotopy equivalence of $\mathcal{A}/\mathcal{G}$ and $\Omega^3(SU(N))$.

In this Note we discuss the index of D on the unit ball with the boundary condition on S^3 given by W_A and prove that it is equal to the index of D_A (on S^4). The influence of the vector potential is absorbed in the boundary condition (see Theorems 4 and 5).

This result follows from the Atiyah–Patodi–Singer Theorem (see [1]) in the case of a product metric structure near the *equator* S^3. Even in our classical situation, however, the metric is not product, but depends on the normal coordinate. Our proof does not use the Atiyah–Patodi–Singer Theorem and covers this more general situation.

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1. Preliminaries on the Dirac operator

We follow the presentation from [4]. Let $\Delta^\pm$ denote two half–spin representations of $Spin(4)$, $\dim \Delta^\pm = 2$. Let $S = \mathbf{C}^2 \times \Delta$ be the spinor bundle on $\mathbf{C}^2$. The corresponding bundle S^+ (resp. S^-) is called the *even* (resp. *odd*) *spinor bundle.* The *(total) Dirac operator* is defined by

$$\mathcal{D} := c \circ d\,,$$

where $d : S \longrightarrow S \otimes T^*\mathbf{C}^2 \simeq S \otimes T\mathbf{C}^2$ denotes the exterior differentiation and $c : S \otimes T\mathbf{C}^2 \longrightarrow S$ the bundle homomorphism coming from the Clifford multiplication. By means of the decomposition $S = S^+ \oplus S^-$ the Dirac operator has the chiral decomposition

$$\mathcal{D} = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} : C^\infty\left(\mathbf{C}^2, S^+ \oplus S^-\right) \longrightarrow C^\infty\left(\mathbf{C}^2, S^+ \oplus S^-\right).$$

The (half) operators D and $D^\dagger$ have the following coordinate expressions

$$D^\dagger = \begin{pmatrix} \frac{\partial}{\partial \bar{z}_1} & \frac{\partial}{\partial \bar{z}_2} \\ -\frac{\partial}{\partial z_2} & \frac{\partial}{\partial z_1} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \frac{\partial}{\partial z_1} & -\frac{\partial}{\partial \bar{z}_2} \\ \frac{\partial}{\partial z_2} & \frac{\partial}{\partial \bar{z}_1} \end{pmatrix}.$$

Let

$$\nu = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}, \qquad \epsilon = -\bar{z}_2 \frac{\partial}{\partial z_1} + \bar{z}_1 \frac{\partial}{\partial z_2}.$$

The *radial vector field* is defined by

$$\frac{\partial}{\partial n} = \frac{1}{2|z|}(\nu + \bar{\nu}), \qquad |z| \neq 0.$$

We also introduce

$$\theta_3 = \frac{1}{2\sqrt{-1}}(\nu - \bar{\nu}), \quad \theta_2 = \frac{1}{2\sqrt{-1}}(\epsilon - \bar{\epsilon}), \qquad \theta_1 = \frac{1}{2}(\epsilon + \bar{\epsilon}).$$

The quartet $\{ \frac{\partial}{\partial n}, \frac{1}{|z|}\theta_i, i = 1, 2, 3 \}$ forms an orthogonal frame on $\mathbf{C}^2 \setminus 0$.

Now the Dirac operator can be expressed in the following way

$$\mathcal{D} = (\frac{\partial}{\partial n}) \cdot \nabla_{(\frac{\partial}{\partial n})} + \sum_{i=1}^{3} (\frac{1}{|z|}\theta_i) \cdot \nabla_{\frac{1}{|z|}\theta_i}, \quad |z| \neq 0,$$

where $\nabla_{\mathbf{v}}$ denotes the covariant differentiation into the direction of $\mathbf{v}$ and $\cdot$ denotes the Clifford multiplication. The second term gives the (*tangential*) Dirac operator on the latitude $B_r = \{|z| = r\}$.

We denote by γ_0 the Clifford multiplication by the radial vector $\frac{\partial}{\partial n}$. Let us notice that γ_0 interchanges the chirality:

$$\gamma_0 : S^+ \oplus S^- \longrightarrow S^- \oplus S^+, \qquad \gamma_0^2 = 1.$$

In particular, γ_0 has the following coordinate expression

$$\gamma_+ = \gamma_0|S^+ = \frac{1}{|z|}\begin{pmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{pmatrix}, \qquad \gamma_- = \gamma_0|S^- = \frac{1}{|z|}\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}.$$

On $\{|z| \neq 0\}$, the Dirac operators D and $D^\dagger$ have the following polar decompositions:

$$D = \gamma_+ \left(\frac{\partial}{\partial n} - \partial\!\!\!/ \right) \qquad \text{and} \qquad D^\dagger = \left(\frac{\partial}{\partial n} + \partial\!\!\!/ + \frac{3}{2|z|} \right) \gamma_-,$$

where the tangential (nonchiral) Dirac operator $\partial\!\!\!/$ is given by

$$\partial\!\!\!/ := -(\gamma_-) \left[\sum_{i=1}^{3} (\frac{1}{|z|}\theta_i) \cdot \nabla_{\frac{1}{|z|}\theta_i} \right] = \frac{1}{|z|} \begin{pmatrix} -\sqrt{-1}\theta_3 & \bar{\epsilon} \\ -\epsilon & \sqrt{-1}\theta_3 \end{pmatrix}.$$

We found the eigenvalues of $\partial\!\!\!/$ in [4] (see also [7]):

THEOREM. *The eigenvalues of $\partial\!\!\!/$ restricted to $B = \{|z| = 1\}$ are*

$$\frac{r}{2}, \qquad -\frac{3+r}{2}; \quad r = 0, 1, 2, \ldots$$

with multiplicity $(r+1)(r+2)$.

2. Static vector potentials

Let M be a domain in $\mathbf{C}^2$ and $E = M \times \mathbf{C}^N$ be a vector bundle. Let A be an $su(N)$–gauge potential on M. Locally A is an $su(N)$–valued 1–form acting on the spinors by the Clifford multiplication c such that

$$A : \Gamma(S^+ \otimes E) \longrightarrow \Gamma(S^- \otimes E).$$

Thus A is given by

$$A(z) = A_1 dz_1 + A_2 dz_2 + A_{\overline{1}} d\overline{z}_1 + A_{\overline{2}} d\overline{z}_2 = \begin{pmatrix} A_1 & -A_{\overline{2}} \\ A_2 & A_{\overline{1}} \end{pmatrix},$$

where the entries A_i etc. are $N \times N$ matrices. A $SU(N)$–gauge transformation acts by the rule

$$(f \cdot A)(z) = f(z)A(z)f(z)^{-1} + f(z)df^{-1}(z).$$

Let $\mathbf{n}^*, \theta_0^*, \epsilon^*, \overline{\epsilon}^*$ denote the dual frame of $\{ \frac{\partial}{\partial n}, \frac{1}{|z|}\theta_3, \frac{1}{|z|}\epsilon, \frac{1}{|z|}\overline{\epsilon} \}$, i.e.

$$\mathbf{n}^* = \frac{1}{|z|}(\nu^* + \overline{\nu}^*), \qquad \theta_3^* = \frac{\sqrt{-1}}{|z|}(\nu^* - \overline{\nu}^*),$$
$$\nu^* = \frac{1}{|z|}(\overline{z}_1 dz_1 + \overline{z}_2 dz_2), \quad \epsilon^* = \frac{1}{|z|}(-z_2 dz_1 + z_1 dz_2).$$

It is convenient to write a 1–form φ on $\mathbf{C}^2 \setminus \{0\}$ in the form

$$\varphi = b_n \mathbf{n}^* + b_3 \theta_3^* + b_+ \epsilon^* + b_- \overline{\epsilon}^*$$

with

$$b_n = \varphi(\frac{\partial}{\partial n}), \quad b_3 = \varphi(\frac{\theta_3}{|z|}), \quad b_+ = \varphi(\frac{\epsilon}{|z|}), \quad b_- = \varphi(\frac{\overline{\epsilon}}{|z|}).$$

Then any vector potential A acting on spinors on $M \setminus \{0\}$ can be written as

$$A = \gamma_+(A_n + A^b) \quad \text{with } A^b = \begin{pmatrix} \sqrt{-1}A_3 & -A_- \\ A_+ & -\sqrt{-1}A_3 \end{pmatrix}.$$

Let $\mathcal{A}_{st}$ denote the space of the vector potentials that are *static* near the equator $B = \{|z| = 1\}$, that is A satisfies the following conditions (1) and (2) in a bi–collar neighbourhood of B:

1. $A_n = \frac{1}{2|z|}\left(z_1 A_1 + z_2 A_2 + \overline{z}_1 A_{\overline{1}} + \overline{z}_2 A_{\overline{2}}\right) = 0$,
2. $A^b(z) = \frac{1}{|z|} A^b(\frac{z}{|z|})$.

Then $\mathcal{A}_{st}$ is isomorphic to the space $\mathcal{A}_3$ of gauge potentials on the 3–sphere B.

The gauge transformation group $\mathcal{G}_3$ is seen to be the mapping group $\Omega^3 G$ with $G := SU(N)$. Let $h \in \Omega^3 G$ and extend it to a collar neighbourhood of B by $h(\frac{z}{|z|})$, which we denote also by h. Then the pure gauge potential $h^{-1}dh$ is static. In addition, we have $h^{-1}dh = (\gamma|S^+)(h^{-1}dh)^b$.

3. Gauge invariance of the spectral projection

Let $D_A : C^\infty(M, S^+ \otimes E) \to C^\infty(M, S^- \otimes E)$ be the Dirac operator coupled with $A \in \mathcal{A}$, i.e. $D_A = D \otimes I_E + 1_S \otimes A$, which we shall abbreviate to $D + A$. The vector potential A is transformed to $A_g = g^{-1}Ag + g^{-1}dg$ under the action of a gauge transformation $g \in \mathcal{G}$, and we have $D_{A_g} = g^{-1}D_A g$. For a static vector potential $A \in \mathcal{A}_{st}$, we have a *polar decomposition* of D_A on the equator B:

$$D_A = D + A = \gamma_+\left(\frac{\partial}{\partial n} - \not{\partial}_A\right), \qquad \not{\partial}_A = \not{\partial} - A^b. \tag{1}$$

Here $\not{\partial}$ is an abbreviation of $\not{\partial} \otimes 1_E$. Let $g \in \mathcal{G}$. We have then $(A_g)^b = g^{-1}A^b g + (g^{-1}dg)^b$, where $(A_g)^b$ is defined by the decomposition

$$D_{A_g} = \gamma_+ \left(\frac{\partial}{\partial n} - \not{\partial} + (A_g)^b \right).$$

We denote by $\Pi_{\geq}$ the orthogonal projection of $L^2(S^3, S^+ \otimes E|S^3)$ onto the subspace spanned by the eigenspinors corresponding to the non–negative eigenvalues of $\not{\partial}$. The corresponding projection defined by the eigensubspace of $\not{\partial}_A = \not{\partial} - A^b$ is denoted by $\Pi_{\geq}(A)$.

PROPOSITION 1. *For any $A \in \mathcal{A}_{st}$ the spectral projection $\Pi_{\geq}(A_g)$ is gauge invariant, i.e.*

$$\Pi_{\geq}(A_g) = g^{-1}\Pi_{\geq}(A)g$$

for any $g \in \Omega^3 G$, which we extend to a bi-collar neighbourhood of B by $g(\frac{z}{|z|})$.

In fact we have

$$\not{\partial}_{A_g} = g^{-1}\not{\partial}_A g$$

for $A \in \mathcal{A}_{st}$ and $g \in \Omega^3 G$. Hence the eigenvalues of $\not{\partial}_{A_g}$ coincide with those of $\not{\partial}_A$.

NOTE . For a non–static vector potential, the operator D_A does not have the representation (1) on the equator. Hence our proof does not cover this more general case. Note that in [1] the authors assume a cylindrical neighbourhood of the boundary with all structures being product near the boundary, so that the polar decomposition is always obtainable in their case.

4. Grassmannian manifold of spinors on S^3

In the following, we show that the index of the gauge coupled Dirac operator D_A on S^4 is equal to that of the Dirac operator $D \otimes 1_E$ on the unit ball $R = \{|z| \leq 1\} \subset \mathbf{C}^2$ equipped with a Grassmannian boundary condition which corresponds to the given gauge potential A.

Here the Dirac operator on S^4 is defined as follows. We consider S^4 as a manifold obtained by patching $\mathbf{C}^2_z$ and $\widehat{\mathbf{C}}^2_w$ together by $w = w(z) = \frac{\bar{z}}{|z|^2}$. Here, $\mathbf{C}^2_z$ and $\widehat{\mathbf{C}}^2_w$ are used as local coordinate neighbourhoods. Then $B = \{|z| = 1\} \simeq S^3$ is the equator.

An even spinor $\varphi \in S^+$ on S^4 is a pair of $\varphi \in C^\infty(\mathbf{C}^2_z, \Delta^+)$ and $\widehat{\varphi} \in C^\infty(\widehat{\mathbf{C}}^2_w, \Delta^-)$ patched together by $\widehat{\varphi} = \overline{|z|^3\gamma\varphi}$. The factor $|z|^3$ is due to the conformal factor of the transformation $w = w(z)$.

In the local coordinate $\widehat{\mathbf{C}}^2_w$ we have

$$D : C^\infty(\widehat{\mathbf{C}}^2_w, \Delta^-) \longrightarrow C^\infty(\widehat{\mathbf{C}}^2_w, \Delta^+).$$

The last equation follows from Theorem 6 and

$$\Pi_{\geq}(A) = h^{-1}\Pi_{\geq}h,$$

which is a consequence of $\not\partial_A = h^{-1}\not\partial h$.

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