QUANTUM COHOMOLOGY AND GEOMETRY

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In these lectures we shall describe some aspects of quantum cohomology theory that are related to integrable systems. Roughly speaking, the following inclusions are already well known:

$$\text{integrable systems} \quad \supset \quad \text{Frobenius manifolds} \quad \supset \quad \text{quantum cohomology}$$

Recent work of Dubrovin ([Du2]) and Givental ([Gi2]) suggests that these three objects are in fact *very closely* related. We shall give an introduction to this area, beginning with Frobenius manifolds in §1, then quantum cohomology in §2. In §3 we shall give an example of how the theory of integrable systems can be applied to quantum cohomology.

There are many recent articles in this area. As a sample of books and fundamental references we recommend [Mc-Sa], [Du1], [Gi1], [Co-Ka] and [Ma]. The author's survey articles [Gu1], [Gu2] may also be consulted. Section §2 is in fact a shortened version of part of [Gu1].

§1 Frobenius manifolds

Consider a triple $(\mathbf{R}^n, g, \omega)$, where, for any $t \in \mathbf{R}^n$, $g_t(\ ,\ )$ is a nondegenerate symmetric bilinear form on $\mathbf{R}^m$, and ω_t is a linear map $\mathbf{R}^m \to \mathrm{End}(\mathbf{R}^m)$. We assume the following compatibility condition:

$$\text{(F)} \qquad g(x, \omega(y)(z)) \text{ is symmetric in } x, y, z.$$

It is convenient to introduce the "product notation" $\omega(x)(y) = x \circ y$ (more accurately, $\omega_t(x)(y) = x \circ_t y$, but we usually omit t). Condition (F) implies that $\circ$ is commutative (but it is not necessarily associative). We denote by ∇ the Levi-Cività connection associated to g (regarding g as a not necessarily positive definite Riemannian metric). This is a connection in the trivial bundle $\mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$.

Definition. $(\mathbf{R}^n, g, \omega)$ *is a Frobenius structure if the following conditions hold:*

(1) $\circ_t$ *is associative, for all* $t \in \mathbf{R}^n$

(2) $d\omega = 0$

(3) ∇ *is flat, i.e. has zero curvature*

When $n = m$, this definition says, essentially, that "$\mathbf{R}^n$ is a Frobenius manifold" in the sense of [Du1].

The integrability conditions (1), (2), (3) (together with (F) above) are very complicated; a practical approach is to assume *some* of them and then study the system of p.d.e. which is equivalent to the *remaining* conditions. A particular solution — or rather, its underlying Frobenius structure — will generally appear in more than one such way. Thus, Frobenius structures have the potential to relate quite different systems of p.d.e., and several examples of this phenomenon can be found in [Du1].

The situation relevant to quantum cohomology theory is where $g_t(\ ,\ ) = (\ ,\ )$, a "constant" metric on $\mathbf{R}^n$, from which it follows that $\nabla = d$ (the standard flat connection). From now on we shall assume that g is of this form. It is easy to see that the remaining integrability conditions (1) and (2) are equivalent to

(1) $\omega \wedge \omega = 0$

(2) $(x \circ y, z) = xyzS$ for some "potential function" $S : \mathbf{R}^n \to \mathbf{R}$.

(Here, the notation zS means the derivative of the function S by the constant vector field z, i.e. $zS = dS(z)$.)

These integrability conditions can be rephrased further as follows:

Proposition. $(\mathbf{R}^n, (\ ,\ ), \omega)$ *is a Frobenius structure if and only if the connection* $d + \lambda\omega$ *is flat for all* $\lambda \in \mathbf{R}$.

Proof. The connection $d + \lambda\omega$ is flat if and only if $d(\lambda\omega) + (\lambda\omega)\wedge)\lambda\omega) = 0$ for all λ, i.e. if and only if $d\omega = \omega \wedge \omega = 0$. $\square$

This "zero curvature formulation" justifies the terminology "integrable" and provides a link with the theory of integrable systems. In fact some authors call any p.d.e. which admits a zero curvature formulation an integrable p.d.e.

To obtain a reasonably specific p.d.e. it is necessary to make further restrictions. For example, if we *assume* condition (2), then condition (1) is equivalent to the well known WDVV equations for S. Our point of view in §3 will be exactly the opposite: we assume (1) and investigate (2), where ω comes from the (commutative associative) quantum cohomology algebra $QH^*(M)$ of a Kähler manifold M.

From the point of view of the additive structure, $QH^*(M)$ is the same as the cohomology algebra $H^*(M)$, but $QH^*(M)$ has a family of "quantum products" $\circ_t$ with $t \in H^*(M)$ which may be considered as a deformation of the cup product on $H^*(M)$. Unless otherwise stated, we take complex coefficients for all cohomology groups. Thus, $H^*(M)$ ($\cong \mathbf{C}^{s+1}$, say) acquires the structure of a (complex analytic) Frobenius manifold, although in this article we restrict the parameter t of $\circ_t$ to be an element of $H^2(M)$ ($\cong \mathbf{C}^k$, say). The fact that the quantum product gives a Frobenius structure is a consequence of (nontrivial) properties of Gromov-Witten invariants, as we shall explain in the next section.

A much simpler example is provided by the ordinary (cup) product on $H^*(M)$. Associativity is built into the definition of cup product, and $d\omega = 0$ for the trivial reason that ω is independent of t. The Frobenius condition $(xy, z) = (y, xz)$ follows from the fact that both sides are equal to the evaluation of xyz on M.

§2 Quantum cohomology

Let M be a simply connected (and compact, connected) Kähler manifold, of complex dimension n. We shall assume that the integral cohomology of M is even dimensional and torsion-free, i.e. that

(A1) $\quad H^*(M;\mathbf{Z}) \;=\; \bigoplus_{i=0}^{n} H^{2i}(M;\mathbf{Z}) \;\cong\; \mathbf{Z}^{s+1}.$

The role of the vector space of §1 will be played by $H^*(M) = H^*(M;\mathbf{C}) \cong \mathbf{C}^{s+1}$. There is a family of product operations $\circ_t$ on this vector space, parametrized by $t \in H^2(M) \cong \mathbf{C}^k$ called the *quantum product.* In this section we shall define $\circ_t$ in terms of certain *Gromov-Witten invariants* $\langle A|B|C\rangle_D$.

We need some notation from ordinary cohomology theory. Let

$$\mathrm{PD} : H^i(M;\mathbf{Z}) \to H_{2n-i}(M;\mathbf{Z})$$

be the Poincaré duality isomorphism. This may be defined as the map which sends a cohomology class x to the "cap product" of x with the fundamental class of M. As far as possible we shall use the notation

$$a, b, c, \ldots \;\in\; H^*(M;\mathbf{Z}) \quad (\subseteq H^*(M;\mathbf{C})\ )$$

for cohomology classes, and we write $|a|, |b|, |c|, \ldots$ for the degrees of $a, b, c, \ldots$. We shall write

$$A = \mathrm{PD}(a), B = \mathrm{PD}(b), C = \mathrm{PD}(c), \ldots \in H_*(M; \mathbf{Z}) \quad (\subseteq H_*(M; \mathbf{C}) \ )$$

for the Poincaré dual homology classes, and $|A|, |B|, |C|, \ldots$ for their degrees.

Let

$$\langle \ , \ \rangle : H^i(M; \mathbf{Z}) \times H_i(M; \mathbf{Z}) \to \mathbf{Z}$$

denote the Kronecker (or "evaluation") pairing; we use the same notation for the extended pairing

$$\langle \ , \ \rangle : H^*(M; \mathbf{Z}) \times H_*(M; \mathbf{Z}) \to \mathbf{Z}$$

(thus, $\langle a, B \rangle$ is zero whenever $|a| \neq |B|$). Since there is no torsion, both these pairings are nondegenerate.

The role of the bilinear form of §1 will be played by the ($\mathbf{C}$-linear extension of the) "intersection pairing", which is defined by

$$(\ , \ ) : H^*(M; \mathbf{Z}) \times H^*(M; \mathbf{Z}) \to \mathbf{Z}, \quad (a, b) = \langle ab, M \rangle.$$

On the right hand side, M denotes the fundamental homology class of the manifold M. It is an element of $H_{2n}(M; \mathbf{Z})$, and its Poincaré dual cohomology class — the identity element of the cohomology algebra — will be denoted by $1 \in H^0(M; \mathbf{Z})$ (an exception to our notational convention for cohomology classes!). The homology class represented by a single point of M will be denoted by $Z \in H_0(M; \mathbf{Z})$, and its Poincaré dual cohomology class will be denoted in accordance with our convention by z.

Now, the well known duality between the cup product and the Kronecker pairing may be expressed by the formula

$$\langle ab, M \rangle = \langle a, B \rangle = \langle b, A \rangle.$$

From the nondegeneracy of the Kronecker pairing (in this situation), it follows that the intersection pairing $(\ , \ )$ is nondegerate.

The cup product on $H^*(M; \mathbf{Z})$ may be specified in terms of its "structure constants". To do this, we choose generators as follows:

$$H_*(M; \mathbf{Z}) = \bigoplus_{i=0}^{s} \mathbf{Z} A_i$$

$$H^*(M; \mathbf{Z}) = \bigoplus_{i=0}^{s} \mathbf{Z} a_i$$

and we define "Kronecker dual" cohomology classes $a_0^c, \ldots, a_s^c$ (i.e. the dual basis with respect to $(\ ,\ )$) by $a_i a_j^c = \delta_{ij} z$. Then for any i, j we have

$$a_i a_j = \sum_{\{\alpha \ \mid\ |a_\alpha| = |a_i| + |a_j|\}} \lambda_\alpha^{i,j} a_\alpha^c$$

for some $\lambda_1^{i,j}, \ldots, \lambda_s^{i,j} \in \mathbf{Z}$. These structure constants are given by

$$\lambda_k^{i,j} = \langle a_i a_j a_k, M \rangle.$$

Observe that if $\lambda_k^{i,j} \neq 0$ then the numerical condition

$$|a_i| + |a_j| + |a_k| = 2n$$

must be satisfied.

Definition. *For cohomology classes a, b, c we define $\langle A|B|C\rangle_0 = \langle ab, C\rangle = \langle abc, M\rangle$.*

Giving all the structure constants is the same thing as giving all "triple products" $\langle A|B|C\rangle_0$.

The quantum product will be determined by a larger family of triple products denoted by $\langle A|B|C\rangle_D$, where D varies in $H_2(M; \mathbf{Z})$. (These are the Gromov-Witten invariants $\psi_{3,0}^D(A \otimes B \otimes C)$.) We shall define these new triple products later; for the moment we shall just assume that

(A2) $\langle A|B|C\rangle_D \in \mathbf{Z}$ is defined for any $A, B, C \in H_*(M; \mathbf{Z})$, $D \in H_2(M; \mathbf{Z})$. Moreover, $\langle A|B|C\rangle_D \in \mathbf{Z}$ is linear and symmetrical in A, B, C.

The quantum product is defined as follows:

Definition. $\langle a \circ_t b, C\rangle = \sum_{D \in H_2(M;\mathbf{Z})} \langle A|B|C\rangle_D \, e^{\langle t, D\rangle}$.

We assume that

(A3) in the above definition, the sum on the right hand side is finite.

Observe that as "$t \to -\infty$" the right hand side converges to $\langle A|B|C\rangle_0$; hence $a \circ_t b$ converges to the cup product ab. In this sense, the quantum product is a deformation of the cup product.

The main result concerning this quantum product is:

Theorem. *The triple* ($H^2(M;\mathbf{C})$, (,) , $\circ_t$) *defines a Frobenius structure.*

There is a modification of the quantum product, which is a product operation

$$\circ : H^*(M;\mathbf{C}) \times H^*(M;\mathbf{C}) \to H^*(M;\mathbf{C}) \otimes \Lambda$$

where Λ is the group algebra $\mathbf{C}[H_2(M;\mathbf{Z})]$. Formally, an element of Λ is a finite sum $\sum_i \lambda_i q^{D_i}$, where $\lambda_i \in \mathbf{C}$, $D_i \in H_2(M;\mathbf{Z})$, and where the symbols q^D are multiplied in the obvious way, i.e. $q^D q^E = q^{D+E}$. The definition is:

Definition. $a \circ b = \sum_{D \in H_2(M;\mathbf{Z})} (a \circ b)_D \, q^D$, *where* $(a \circ b)_D$ *is defined by* $\langle (a \circ b)_D, C \rangle = \langle A|B|C \rangle_D$ *for all* $C \in H_*(M;\mathbf{C})$.

We shall also refer to this operation as the quantum product. By our assumption (A3), the sum on the right hand side of the definition of $a \circ b$ is finite.

It is convenient to define a grading on the algebra $H^*(M;\mathbf{C}) \otimes \Lambda$ by defining

$$|aq^D| = |a| + 2\langle c_1 TM, D \rangle.$$

We shall assume that the quantum products preserve this grading, i.e. that $|a \circ b| = |a| + |b|$. It follows that $|(a \circ b)_D| = |a| + |b| - 2\langle c_1 TM, D \rangle$. Hence, if $\langle A|B|C \rangle_D \neq 0$, then the numerical condition $|a| + |b| - 2\langle c_1 TM, D \rangle = |C|$ must be satisfied, i.e.

$$|a| + |b| + |c| = 2n + 2\langle c_1 TM, D \rangle.$$

The geometrical meaning of this numerical condition will become clear later. We write $QH^i(M;\mathbf{C}) = \{x \in QH^*(M;\mathbf{C}) \mid |x| = i\}$.

One further piece of notation will be useful. In all of our examples we shall choose an identification $H_2(M;\mathbf{Z}) \cong \mathbf{Z}^r$ (for some $r \geq 1$). Having made this choice, we write $D = (s_1, \dots, s_r)$, and $q^D = q_1^{s_1} \dots q_r^{s_r}$.

Before defining the Gromov-Witten invariants $\langle A|B|C \rangle_D$, we shall begin by reviewing a special case, namely the definition of the triple product $\langle A|B|C \rangle_0 = \langle ab, C \rangle = \langle abc, M \rangle$ in ordinary cohomology.

The naive definition is

$$\langle A|B|C \rangle_0 \ = \ |\tilde{A} \cap \tilde{B} \cap \tilde{C}|,$$

where the right hand side means the number of points (counted with multiplicity) in the intersection $\tilde{A} \cap \tilde{B} \cap \tilde{C}$, where $\tilde{A}, \tilde{B}, \tilde{C}$ are suitable representatives of the homology classes A, B, C. In certain situations this definition is "correct", in the sense that it gives the usual triple product $\langle abc, M\rangle$. (Similarly, $\langle ab, M\rangle$ may be defined naively as $|\tilde{A}\cap\tilde{B}|$.) For example, in the complex algebraic category, the definition is correct whenever there exist representative algebraic subvarieties $\tilde{A}, \tilde{B}, \tilde{C}$ whose intersection is finite (or empty). Thus, whenever we are lucky enough to find such representatives, we can calculate $\langle A|B|C\rangle_0$.

The most famous example where this method works is the case $M = Gr_k(\mathbf{C}^n)$ (Schubert calculus). Here all the generators of the homology groups are representable by algebraic cycles (Schubert varieties), and for any three such generators a, b, c satisfying the condition $|a| + |b| + |c| = \dim M$ there exist representatives whose intersection is finite (or empty).

We wish to define $\langle A|B|C\rangle_D$ as a certain intersection number. From the previous discussion, it should have the properties

(a) $D = 0 \implies \langle A|B|C\rangle_D = \langle abc, M\rangle$

(b) $\langle A|B|C\rangle_D \neq 0 \implies |a| + |b| + |c| = 2n + 2\langle c_1 TM, D\rangle$.

(Actually, (a) will be obvious from the definition, but (b) will have the status of an assumption.)

The naive definition is

$$\langle A|B|C\rangle_D = |\,\mathrm{Hol}_D^{\tilde{A},p} \cap \mathrm{Hol}_D^{\tilde{B},q} \cap \mathrm{Hol}_D^{\tilde{C},r}\,|$$

where

$$\mathrm{Hol}_D^{\tilde{A},p} = \{\text{holomorphic maps } f : \mathbf{C}P^1 \to M \mid f(p) \in \tilde{A} \text{ and } [f] = D\}$$

and where $\tilde{A}$ is a representative of the homology class A. The points p, q, r are three distinct basepoints in $\mathbf{C}P^1$. The notation $[f]$ denotes the homotopy class of f, which is an element of $\pi_2(M) \cong H_2(M;\mathbf{Z})$.

We *assume* that there exist $\tilde{A}, \tilde{B}, \tilde{C}$ such that the above intersection is finite (or empty); this is essentially the meaning of assumption (A2) of the previous section. More fundamentally, we shall make the following three assumptions:

(A2a) $\mathrm{Hol}_D^{M,p}$ is a (smooth) complex manifold, of complex dimension $n + \langle c_1 TM, D\rangle$. This will be denoted more briefly by Hol_D.

(A2b) $\mathrm{Hol}_D^{\tilde{A},p}$ is a complex submanifold (or subvariety) of $\mathrm{Hol}_D^{M,p}$, and the complex codimension of $\mathrm{Hol}_D^{\tilde{A},p}$ in $\mathrm{Hol}_D^{M,p}$ is equal to the complex codimension of $\tilde{A}$ in M.

(A2c) If $\tilde{A}, \tilde{B}, \tilde{C}$ intersect transversely, then so do $\mathrm{Hol}_D^{\tilde{A},p}, \mathrm{Hol}_D^{\tilde{B},q}, \mathrm{Hol}_D^{\tilde{C},r}$.

We shall not discuss the extent to which (A2) is equivalent to (A2a), (A2b), (A2c), nor the extent to which these conditions are true. (Certainly all four assumptions hold for homogeneous Kähler manifolds.) However, a brief comment on the origin of the numerical expressions in (A2a) and (A2b) may be helpful. First, the tangent space at $f \in \mathrm{Hol}_D^{M,p}$ — assuming that $\mathrm{Hol}_D^{M,p}$ is a manifold — may be identified with the space of holomorphic sections of the bundle f^*TM. By the Riemann-Roch theorem, the complex dimension of this vector space is $n + \langle c_1 TM, D\rangle$, as stated in (A2a). Regarding (A2b), this would follow from the commutative diagram

$$\begin{array}{ccc} \mathrm{Hol}_D^{\tilde{A},p} & \xrightarrow{\subseteq} & \mathrm{Hol}_D^{M,p} \\ \downarrow & & \downarrow \\ \tilde{A} & \xrightarrow{\subseteq} & M \end{array}$$

in which the vertical maps are given by $f \mapsto f(p)$, whenever the evaluation map $\mathrm{Hol}_D^{M,p} \to M$ is a locally trivial fibre bundle (as it will be when M is homogeneous, for example).

It is important to bear in mind that the space Hol_D is finite-dimensional and algebraic, so there is no question of infinite-dimensional analysis here. The technical problems in giving a rigorous definition arise from the noncompactness of Hol_D, and the fact that the transversality condition may not be satisfied. We conclude with a very brief discussion of what is needed. First, a very general definition of $\langle A|B|C\rangle_D$ (and hence of the quantum product) is possible under the assumption that the (connected simply connected Kähler—or even merely symplectic) manifold M is "positive" in some sense, for example that $\langle c_1 TM, D\rangle > 0$ for each homotopy class $D \in \pi_2(M)$ which contains a holomorphic map $\mathbf{C}P^1 \to M$. A Fano manifold — that is, a manifold M for which the cohomology class $c_1 TM$ can be represented by a Kähler 2-form — is automatically positive in this sense. It can be shown that the quantum product is commutative and associative, when M is positive. So far, however, there is no guarantee that the Gromov-Witten invariant $\langle A|B|C\rangle_D$ can be *computed* by the naive formula given earlier in this section. For this, one needs an additional assumption, for example that M is "convex" in the sense that $H^1(\mathbf{C}P^1, f^*TM) = 0$ for all holomorphic $f : \mathbf{C}P^1 \to M$. Convexity implies in particular that Hol_D is a manifold, and that it has the "expected" dimension $n + \langle c_1 TM, D\rangle$. Homogeneous Kähler manifolds are convex, for example.

The series defining the quantum product will in general contain infinitely many powers of q (both positive and negative). However, if there exist $D_1, \dots, D_r \in \pi_2(M)$ such that all holomorphically representable classes D are of the form $\sum_{i=1}^r n_i D_i$ with $n_i \geq 0$, then it follows from the positivity condition that each such series contains only a finite number of terms $q^D = q_1^{s_1} \dots q_r^{s_r}$, and that $s_i \geq 0$ in each case. Any simply connected homogeneous Kähler manifold satisfies this condition, as well as positivity and convexity.

§3 An application of integrable systems to quantum cohomology

Let $K = \mathbf{R}[q_1, \ldots, q_r]$ be the ring of polynomials in $q_1, \ldots, q_r$ with real coefficients. Depending on the context, we regard q_i either as a formal variable or as a function $t \mapsto e^{t_i}$ where $t = (t_1, \ldots, t_r) \in \mathbf{R}^r$. We introduce the notation $\partial_i = \frac{\partial}{\partial t_i} = q_i \frac{\partial}{\partial q_i}$, and define D to be the ring of differential operators generated by $\partial_1, \ldots, \partial_r$ with coefficients in K. Let $M = D/(D_1, \ldots, D_u)$ be a cyclic D-module (a left module over D, generated by the constant differential operator 1), where $(D_1, \ldots, D_u)$ means the left ideal generated by differential operators $D_1, \ldots, D_u$. We shall assume that M is holonomic, of rank $s+1$. This implies in particular that, as a K-module, M is isomorphic to the direct sum of $s+1$ copies of K.

The D-module M is an algebraic version of the system of partial differential equations $D_1 f = \cdots = D_u f = 0$. Here, f belongs to a given function space F, but M is of course independent of F (and this is its advantage). To say that M is holonomic is to say that the system is "maximally overdetermined"; in particular its solution space is finite dimensional. More precisely, the vector space $\mathrm{Hom}_D(M, F)$ is called the *solution space of* M, and this is isomorphic to the usual solution space $\{f \in F \mid D_1 f = \cdots = D_u f = 0\}$ of the system: to a solution f there corresponds the D-module homomorphism $M \to F$ given by $P \mapsto Pf$ (for any $P \in D$). The solution space (in either sense) is a real vector space of dimension $s+1$.

We shall review briefly the equivalence between holonomic D-modules and flat connections. Let us choose differential operators $P_0, \ldots, P_s$ such that the equivalence classes $[P_0], \ldots, [P_s]$ form a K-module basis of M. Without loss of generality we may take $P_0 = 1$. With respect to this basis we define matrices $\Omega_i = (\Omega^i_{kj})_{0 \le k,j \le s}$ by

$$[\partial_i P_j] = \sum_{k=0}^{s} \Omega^i_{kj} [P_k],$$

and we define $\Omega = \sum_{i=1}^{r} \Omega_i dt_i$, a 1-form with values in the space $\mathrm{End}(\mathbf{R}^{s+1})$ of real $(s+1) \times (s+1)$ matrices.

The formula $\nabla = d + \Omega$ defines a connection in the trivial vector bundle $\mathbf{R}^r \times \mathbf{R}^{s+1}$, where $\mathbf{R}^{s+1}$ is identified with the vector space spanned by $[P_0], \ldots, [P_s]$. Namely, $\nabla_{\partial_i}[P_j] = \sum_{k=0}^{s} \Omega^i_{kj}[P_k]$, and more generally for any section $\sum_{j=0}^{s} y_j [P_j]$ of this bundle,

$$\nabla_{\partial_i}\Big(\sum_{j=0}^{s} y_j [P_j]\Big) = \sum_{j=0}^{s} \partial_i y_j [P_j] + \sum_{j=0}^{s} y_j \nabla_{\partial_i}[P_j] = \sum_{j=0}^{s} \partial_i y_j [P_j] + \sum_{j,k=0}^{s} y_j \Omega^i_{kj} [P_k].$$

Proposition. *The connection ∇ is flat.*

Proof. By definition we have $\nabla_{\partial_i}\nabla_{\partial_j} = \nabla_{\partial_j}\nabla_{\partial_i}$ (since $\partial_i\partial_j = \partial_j\partial_i$). It follows that the curvature tensor of ∇ is zero. $\square$

In terms of the 1-form Ω, the zero curvature condition is $d\Omega + \Omega\wedge\Omega = 0$. This condition may be verified directly by computing both sides of the equation $\partial_i\partial_j[P_k] = \partial_j\partial_i[P_k]$, thus giving an alternative proof of the proposition.

Proposition. *We have an isomorphism of vector spaces*

$$\mathrm{Hom}_D(M,F) \longrightarrow \{\textit{covariant constant sections of } \nabla^*\}, \quad f \longmapsto \begin{pmatrix} P_0 f \\ \vdots \\ P_s f \end{pmatrix}$$

where ∇^* is the dual connection to ∇.

Proof. On the left hand side, f is regarded as the D-module homomorphism $P \mapsto Pf$, whereas on the right hand side f is a solution of the system $D_1 f = \cdots = D_u f = 0$. The dual connection is defined by $(\nabla^*_{\partial_i}[P_j]^*)[P_\alpha] = -[P_j]^*(\nabla_{\partial_i}[P_\alpha])$ where $[P_0]^*,\dots,[P_s]^*$ is the dual basis to $[P_0],\dots,[P_s]$. The column vector in the statement of the proposition refers to the section $\sum_{j=0}^s (P_j f)[P_j]^*$. A section $\sum_{j=0}^s y_j[P_j]^*$ is covariant constant if

$$\begin{aligned}(\nabla^*_{\partial_i}\sum_{j=0}^s y_j[P_j]^*)[P_\alpha] &= (\sum_{j=0}^s \partial_i y_j[P_j]^* + \sum_{j=0}^s y_j\nabla^*_{\partial_i}[P_j]^*)[P_\alpha] \\ &= \partial_i y_\alpha - \sum_{j=0}^s y_j([P_j]^*\sum_{\beta=0}^s \Omega^i_{\beta,\alpha}[P_\beta]) \\ &= \partial_i y_\alpha - \sum_{j=0}^s y_j\Omega^i_{j,\alpha}\end{aligned}$$

is zero for all α. For any $f \in \mathrm{Hom}_D(M,F)$, we have to verify that $y_\alpha = P_\alpha f$ defines a covariant constant section. But this follows immediately from the formula $[\partial_i P_\alpha] = \sum_{k=0}^s \Omega^i_{j\alpha}[P_j]$ defining Ω. The map in question is therefore a well defined, linear, map. To prove that it is an isomorphism, we observe that the kernel is zero (because $P_0 f = f$), and that $\dim\mathrm{Hom}_D(M,F) = s+1$ by the holonomy assumption. $\square$

The proof generalizes the well known elementary construction of a system of first order o.d.e. equivalent to a higher order o.d.e. Here we construct the system $\partial_i y_\alpha = \sum_{j=0}^s y_j\Omega^i_{j\alpha}$ of first order p.d.e. equivalent to the system $D_1 f = \cdots = D_u f = 0$ of higher order p.d.e (a construction which depends on the choice of the basis $[P_0],\dots,[P_s]$). It should be noted that the solution space of the first order system is associated to the *dual* connection $\nabla^* = d - \Omega^t$.

Conversely, given a flat connection (and hence a system of first order p.d.e.), it is possible to construct a cyclic holonomic D-module (and hence a system of higher order p.d.e.). A choice of cyclic generator is involved here, so the procedure is not canonical. Of course the passage from a D-module to a connection form is not canonical either.

Since the connection $d - \Omega^t$ is flat, there exist covariant constant sections $H_0, \dots, H_s$ which are linearly independent at each point of $\mathbf{R}^r$. Representing these sections by column vectors, as above, let us introduce

$$H = \begin{pmatrix} | & & | \\ H_0 & \dots & H_s \\ | & & | \end{pmatrix}$$

i.e. the "fundamental solution matrix" of the first order system. By definition we have $\Omega^t = dHH^{-1}$. Up to multiplication on the right by a constant invertible matrix, this equation determines H uniquely.

Equivalently, if $f_0, \dots, f_s$ are a basis of solutions of the higher order system $D_1 f = \dots = D_u f = 0$, and if $J = (f_0, \dots, f_s)$ is regarded as a row vector, then

$$H = \begin{pmatrix} - & P_0 J & - \\ & \vdots & \\ - & P_s J & - \end{pmatrix}$$

satisfies $\Omega^t = dHH^{-1}$.

A standard technique in the theory of D-modules is to study the transformation

$$D \to K[b_1, \dots, b_r], \quad \partial_i \mapsto b_i,$$

from the non-commutative algebra D to a (commutative) polynomial algebra. A differential operator P (written in "standard form" with the coefficients to the left of the derivatives) maps to a polynomial $\tilde{P}$. The D-module $M = D/(D_1, \dots, D_u)$ is transformed to a K-module $\tilde{M} = K[b_1, \dots, b_r]/(\tilde{D}_1, \dots, \tilde{D}_u)$.

The associated flat connection ∇ is transformed to a connection $\tilde{\nabla}$ *but this connection is not in general flat.* In more detail, we have $\nabla = d + \sum_{i=1}^r \Omega_i dt_i$ where Ω_i is the matrix representing the action of the differential operator ∂_i, and $\tilde{\nabla} = d + \sum_{i=0}^r \tilde{\Omega}_i dt_i$ where $\tilde{\Omega}_i$ is the matrix representing the action of the (zero-th order) operator b_i. As explained earlier, the fact that $\partial_i \partial_j = \partial_j \partial_i$ leads to the flatness condition $d\Omega + \Omega \wedge \Omega = 0$. However, the condition $b_i b_j = b_j b_i$ says only that $\tilde{\Omega} \wedge \tilde{\Omega} = 0$. The exterior derivative $d\tilde{\Omega}$ is not in general zero.

Now we turn to quantum cohomology. Our goal is to characterise a D-module arising from the quantum cohomology algebra in a certain situation, namely where the ordinary cohomology algebra is of the form

$$H^*(M) = \mathbf{C}[b_1, \dots, b_r]/(R_1, \dots, R_u)$$

where $b_1, \dots, b_r$ are additive generators of $H^2(M)$ and $R_1, \dots, R_u$ are certain relations (polynomials in $b_1, \dots, b_r$). By general principles it follows that the (small) quantum cohomology algebra is of the form

$$QH^*(M) = K[b_1, \dots, b_r]/(\mathcal{R}_1, \dots, \mathcal{R}_u)$$

where $K = \mathbf{C}[q_1, \dots, q_r]$ and each $\mathcal{R}_i$ is a "q-deformation" of R_i. The variables $q_1, \dots, q_r$ here may be considered either as formal variables or as functions

$$q_i : H^2(M) = \bigoplus_{j=1}^{r} \mathbf{C} b_j \longrightarrow \mathbf{C}, \quad t = \sum_{j=1}^{r} t_j b_j \longmapsto e^{t_i},$$

as explained in §2.

Let us choose an additive basis $c_0, \dots, c_s$ of the vector space $QH^*(M)$ (thus, each c_i is a polynomial in $b_1, \dots, b_r$). For $i = 1, \dots, r$, let ω_i denote the matrix of multiplication by b_i on the vector space $QH^*(M)$ with respect to the basis[1] $c_0, \dots, c_s$. Let $\omega = \sum_{i=1}^{r} \omega_i dt_i$. To make contact with the theory of §1, we introduce the ring D^h of differential operators generated by $\partial_1, \dots, \partial_r$ with coefficients in $K[h]$, where h is a nonzero complex parameter. Each relation $\mathcal{R}_i$ gives rise to a differential operator D_i^h, obtained by replacing $b_1, \dots, b_r$ by $h\partial_1, \dots, h\partial_r$.

Definition. *The cyclic D-module $M^h = D^h/(D_1^h, \dots, D_u^h)$ is called the D-module associated to the quantum cohomology algebra $QH^*(M)$.*

Let $\Omega^h = \sum_{i=1}^{r} \Omega_i^h dt_i$ be the $\mathrm{End}(\mathbf{C}^{s+1})$-valued 1-form defined using the D-module M^h as in §1, i.e. Ω_i^h is the matrix of the action of ∂_i on the $K[h]$-module M^h.

From the differentiation formula

$$h\partial_i e^{t_j} f = \begin{cases} e^{t_j} h\partial_i f & \text{if } i \neq j \\ e^{t_j} h\partial_i f + h e^{t_j} f & \text{if } i = j \end{cases}$$

[1]Strictly speaking, we should write $[c_0], \dots, [c_s]$, but usually we omit the brackets in order to simplify the notation.

it follows that Ω^h and ω are related by

$$\Omega^h = \frac{1}{h}\omega + \theta$$

for some θ (where both θ and ω are independent of h).

Proposition. *The connection $\nabla^h = d + \Omega^h$ is gauge-equivalent to a connection of the form $d + \frac{1}{h} Q\omega Q^{-1}$, for some map $Q : H^2(M) \to GL(H^*(M))$.*

Proof. A gauge transformation by $G : H^2(M) \to GL(H^*(M))$ transforms Ω^h into $G^{-1}\Omega^h G + G^{-1}dG$ (this amounts to replacing F by FG, where $\Omega^h = F^{-1}dF$). To find a suitable gauge transformation, we write out the flatness condition $d\Omega^h + \Omega^h \wedge \Omega^h = 0$. With $\Omega^h = \frac{1}{h}\omega + \theta$, we obtain

$$\frac{1}{h}d\omega + d\theta + \frac{1}{h^2}\omega \wedge \omega + \frac{1}{h}(\omega \wedge \theta + \theta \wedge \omega) + \theta \wedge \theta = 0.$$

This holds for all h, so we obtain $\omega \wedge \omega = 0$, $d\omega + \omega \wedge \theta + \theta \wedge \omega = 0$, and $d\theta + \theta \wedge \theta = 0$. The last condition is equivalent to the existence of a map Q such that $\theta = Q^{-1}dQ$. Taking $G = Q^{-1}$ gives the gauge transformation

$$\Omega^h \mapsto Q\Omega^h Q^{-1} + Qd(Q^{-1}) = \frac{1}{h}Q\omega Q^{-1} + Q\theta Q^{-1} - dQQ^{-1} = \frac{1}{h}Q\omega Q^{-1}$$

which has the required effect. □

Corollary. *Let $\hat{c}_i = Q^{-1}c_i$ where Q is as in the proposition. Then $\hat{\theta} = 0$, and the 1-forms $\hat{\omega}$ and $\hat{\Omega}^h$ (computed using the basis $\hat{c}_0, \ldots, \hat{c}_s$) are related by the formula*

$$\hat{\Omega}^h = \frac{1}{h}\hat{\omega}.$$

It follows that $d\hat{\omega} = \hat{\omega} \wedge \hat{\omega} = 0$.

Proof. The change of basis replaces ω by $\hat{\omega} = Q\omega Q^{-1}$, and it replaces Ω^h by

$$\hat{\Omega}^h = Q\Omega^h Q^{-1} + Qd(Q^{-1}) = \frac{1}{h}Q\omega Q^{-1} = \frac{1}{h}\hat{\omega}$$

as explained in the proof of the proposition. Since $d + \Omega^h$ is flat, so is $d + \hat{\Omega}^h$, hence $hd\hat{\omega} + \hat{\omega} \wedge \hat{\omega} = 0$. This is true for all h, so we must have $d\hat{\omega} = \hat{\omega} \wedge \hat{\omega} = 0$. □

Returning to the general theory, we note that the change of basis from the c_i's to the $\hat{c}_i$'s is essentially unique, because the function Q is determined, up to multiplication on the left by a constant matrix, from the equation $\theta = Q^{-1}dQ$. To understand the significance of the distinguished basis we go back to quantum cohomology theory.

Up to this point, we have considered only $QH^*(M)$, ignoring completely the ordinary cohomology algebra $H^*(M)$. But quantum cohomology theory gives much more information than the (isomorphism type of the) algebra $QH^*(M)$: it gives a "quantum product operation" on the vector space $H^*(M)$, and — with respect to a choice of additive basis of $H^*(M)$ — it gives "i-point Gromov-Witten invariants" for $i = 0, 1, 2, \ldots$. The quantum product is determined by the 3-point Gromov-Witten invariants, and it is a q-deformation of the ordinary (cup) product. That is, for any $x, y \in H^*(M)$, there is a quantum product $x \circ y \in H^*(M)$, which has the property $x \circ y = x \cdot y +$ terms involving $q_i = e^{t_i}$, $1 \le i \le r$, where $x \cdot y$ denotes the cup product. The relations $R_1, \ldots, R_u$ are those of the algebra $(H^*(M), \cdot\ )$, while the relations $\mathcal{R}_1, \ldots, \mathcal{R}_u$ are those of the algebra $(H^*(M), \circ\ )$.

The additional information given by the quantum product amounts to a choice of an (additive) isomorphism between $H^*(M)$ and $QH^*(M)$. One way to specify such an isomorphism is to *declare* that the additive basis $c_0, \ldots, c_s$ of $QH^*(M)$ corresponds to the additive basis $c_0|_{q=0}, \ldots, c_s|_{q=0}$ of $H^*(M)$ (assuming of course that these elements are well defined and linearly independent in $H^*(M)$).

Without loss of generality we may assume that $c_0 = c_0|_{q=0} = 1$ and $c_i = c_i|_{q=0} = b_i$ for $i = 1, \ldots, k$. Let us simplify the notation by writing $b_j = c_j|_{q=0}$ for $j = k+1, \ldots, s+1$. In terms of the additive basis $b_0, \ldots, b_s$ of $H^*(M)$, the quantum product is defined by

$$b_i \circ b_j = \sum_{k=0}^{s} a_{ijk} b_k$$

where the a_{ijk} are given by multiplication of the corresponding classes in the algebra $QH^*(M)$, i.e. $c_i c_j = \sum_{k=0}^{s} a_{ijk} c_k$ Clearly the quantum product is determined by giving all such products of basis elements.

It turns out that the 1-forms ω and Ω^h in this situation, i.e. in the presence of a quantum product, have the characteristic property discussed earlier:

Proposition. *Assume that ω and Ω^h arise from the quantum product in the above fashion. In other words, the basis $c_0, \ldots, c_s$ has the property that the "declared" product operation on $H^*(M)$ agrees with the usual quantum product operation defined using 3-point Gromov-Witten invariants. Then $\Omega^h = \frac{1}{h}\omega$, and $d\omega = \omega \wedge \omega = 0$.*

Proof. The equation $\omega \wedge \omega = 0$ is an immediate consequence of the commutativity and associativity of $QH^*(M)$. The equation $d\omega = 0$ would follow from the existence of a function K such that $\omega = dK$. Such a function may be constructed explicitly from

Gromov-Witten invariants (see, for example, [Gu2]). It follows that the connection $d+\frac{1}{h}\omega$ is flat for all values of h, and so is the dual connection $d-\frac{1}{h}\omega^t$. Let H be a fundamental solution matrix for the system $dHH^{-1} = \frac{1}{h}\omega^t$, i.e. the columns of H are a basis of covariant constant sections with respect to $d-\frac{1}{h}\omega^t$. Now, the function $H^\dagger$ whose rows are $P_0^h J, \dots, P_s^h J$ (where J is as in §1 and $\tilde{P}_i^h = c_i$) also satisfies the equation $dH^\dagger {H^\dagger}^{-1} = \frac{1}{h}\omega^t$. Hence $H^\dagger = HA$ for some (constant, invertible) matrix A. It follows that $(\Omega^h)^t = dH^\dagger {H^\dagger}^{-1} = dHH^{-1} = \frac{1}{h}\omega^t$, as required. $\square$

Further applications of this point of view will be discussed elsewhere.

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