HOMOTOPY INVARIANT ALGEBRAIC STRUCTURE

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In this note we give a *pictorial proof* of the statement, which is due to Kontsevich-Soibelman [KS], that the structure of A_∞-category is transferred over homotopy. For A_∞-algebras, Kontsevich-Soibelman [KS] gives a sketch of proof of the corresponding statement, and Akaho [A] gives a complete proof. This note owes much to their ideas.

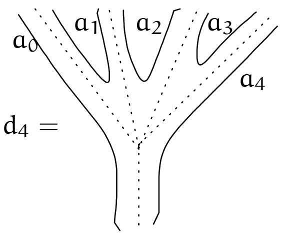
1. A_∞-CATEGORY

Definition 1.1 ([F]). An A_∞-*category* $\mathcal{C}$ consists of

- $\mathrm{Ob}(\mathcal{C})$: a class of 'objects'
- $V(a,b)$: a $\mathbb{Z}$-graded vector space of 'arrows', for each pair $a, b \in \mathrm{Ob}(\mathcal{C})$
- a linear map of degree 1, called *composition map*,

  $$d_k : V(a_o, a_1)[1] \otimes V(a_1, a_2)[1] \otimes \cdots \otimes V(a_{k-1}, a_k)[1] \to V(a_0, a_k)[1],$$

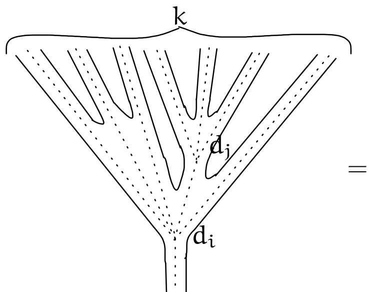
  for each $k \geq 1$ and each $a_0, a_1, \ldots, a_k \in \mathrm{Ob}(\mathcal{C})$,

$d_4 =$ 

$$\begin{array}{c} V(a_0,a_1)[1]\otimes V(a_1,a_2)[1]\otimes V(a_2,a_3)[1]\otimes V(a_3,a_4)[1] \\ \downarrow \\ V(a_0,a_4)[1] \end{array}$$

satisfying the following relation for each $k \geq 1$:

$$\sum_{i+j=k+1} \sum_{1\leq s\leq i} d_i(\mathbb{I}^{\otimes s-1} \otimes d_j \otimes \mathbb{I}^{\otimes i-s}) = 0. \tag{1.1}$$

$\sum_{i+j=k+1} \sum_{1\leq s\leq i}$  $= 0$

The symbol [1] means a shift of degree, that is, for any graded vector space V, $V[1]$ is a graded vector space whose component of degree n is $(V[1])^n = V^{n+1}$. For simplicity, we use the same symbol $\mathbb{I}$ for the identity maps on vector spaces $\{V(a_{i-1}, a_i)[1]\}_{1\leq i\leq k}$.

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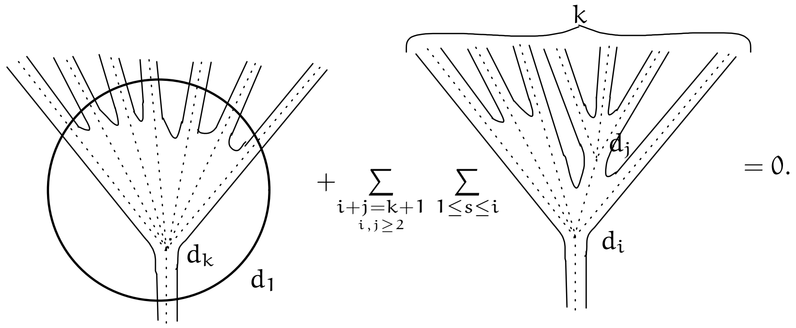
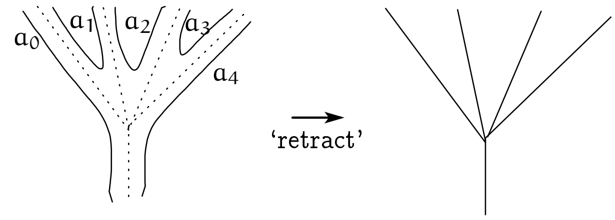
In the sequel, it is convenient to rewrite the relation, picking out parts that i or j equal to 1, as follows:



Example 1.1 (see [D] [L] [M]). Each complex manifold X gives an A_∞-category $\mathcal{C}_X$:

- objects are holomorphic vector bundles on X;
- the space $V(E, F)$ of arrows is $\Omega^{0,\bullet}(\mathrm{Hom}(E, F))$ for each pair of holomorphic vector bundles E, F;
- the composition maps are $d_1 = \bar{\partial}_{\mathrm{Hom}(E,F)}[1]$, $d_2 = \wedge[1]$, and $d_k = 0$ for $k \geq 3$.

Remark 1.1. An A_∞-category with only one object is an A_∞-algebra:



2. SUM OF TREES

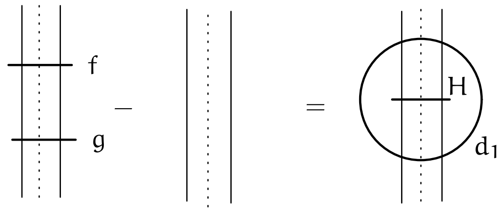
Let $\mathcal{C}$ be an A_∞-category. Suppose we have a family of complexes $\mathcal{C}' = \{V'(a, b)\}_{a,b\in \mathrm{Ob}(\mathcal{C})}$ which is (right) 'homotopic' to $\mathcal{C}$. More precisely, suppose that, for each pair $a, b \in \mathbf{Ob}(\mathcal{C})$, we have

- a $\mathbb{Z}$-graded vector space $V'(a, b)$ with differential d'_1
- a chain map $f : (V(a, b)[1], d_1) \to (V'(a, b)[1], d'_1)$ of degree 0
- a chain map $g : (V'(a, b)[1], d'_1) \to (V(a, b)[1], d_1)$ of degree 0
- a linear map $H : V(a, b)[1] \to V(a, b)[1]$, called *homotopy operator*, of degree -1

satisfying the following relation:

$$g \circ f - \mathbb{I} = Hd_1 + d_1H. \tag{2.1}$$

Pictorially, this relation is



Example 2.1 (see [D] [L] [M]). Let X be a complex manifold. Choosing a Hermitian metric on X and a Hermitian metric on each holomorphic vector bundle on X, we get a family of complexes $\mathcal{C}'_X$ homotopic to the A_∞-category $\mathcal{C}_X$ in the example 1.1:

- for each pair of holomorphic vector bundles E, F on X, a graded vector space $V'(E,F)$ is the space of harmonic forms $\mathbb{H}^{\bullet,0}(\mathrm{Hom}(E,F))$ with differential $d'_1 = 0$
- a chain map $f : \Omega^{\bullet,0}(\mathrm{Hom}(E,F))[1] \to \mathbb{H}^{\bullet,0}(\mathrm{Hom}(E,F))[1]$ is the projection to harmonic forms
- a chain map $g : \mathbb{H}^{\bullet,0}(\mathrm{Hom}(E,F))[1] \to \Omega^{\bullet,0}(\mathrm{Hom}(E,F))[1]$ is the inclusion
- a homotopy operator $H : V(a,b)[1] \to V(a,b)[1]$ is $\bar{\partial}^*_{\mathrm{Hom}(E,F)}G[1]$, where G is the Green operator for the $\bar{\partial}_{\mathrm{Hom}(E,F)}$-Laplacian.

Following Kontsevich-Soibelman [KS], given an A_∞-category $\mathcal{C}$ and a family of complexes $\mathcal{C}' = (\{V'(a,b)\}_{a,b\in \mathbf{Ob}(\mathcal{C})}, d'_1)$ homotopic to $\mathcal{C}$, we construct a linear map

$$d'_k : V'(a_o, a_1)[1] \otimes V'(a_1, a_2)[1] \otimes \cdots \otimes V'(a_{k-1}, a_k)[1] \to V'(a_0, a_k)[1],$$

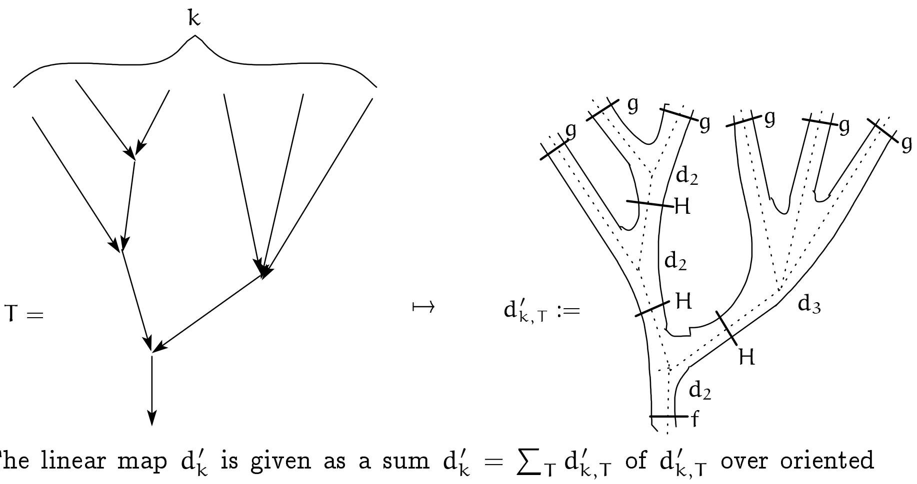
for each $k \geq 2$ and each $a_0, a_1, \ldots, a_k \in \mathrm{Ob}(\mathcal{C})$, from composition maps $\{d_k\}_{k\geq 1}$ of the A_∞-category $\mathcal{C}$, chain maps f, g, and homotopy operator H. In the next section, it is shown that the linear maps $\{d'_k\}_{k\geq 2}$ with d'_1 give an A_∞-category structure on the family of complexes $\mathcal{C}' = \{V'(a,b)\}_{a,b\in \mathbf{Ob}(\mathcal{C})}$. The following two steps give linear maps $\{d'_k\}_{k\geq 2}$:

1. For each oriented planar tree T with k tails and with root,
 a linear map

$$d'_{k,T} : V'(a_o, a_1)[1] \otimes V'(a_1, a_2)[1] \otimes \cdots \otimes V'(a_{k-1}, a_k)[1] \to V'(a_0, a_k)[1],$$

 is given by assigning
 - to each tail vertex, the linear map g
 - to each internal vertex with valency k, the composition map d_k
 - to each internal edge, the homotopy operator H
 - to the root vertex, the linear map f

$T =$ $\mapsto$ $d'_{k,T} :=$



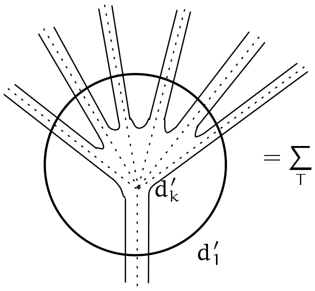
2. The linear map d'_k is given as a sum $d'_k = \sum_T d'_{k,T}$ of $d'_{k,T}$ over oriented planar trees with k tails and with root whose vertex has valency ≥ 2.

3. TRANSFER

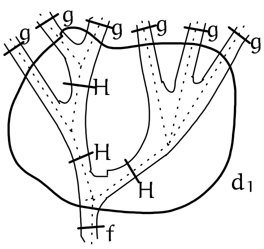
In this section we give a pictorial proof of the following statement which is due to Kontsevich-Soibelman [KS] (see also [A]).

Theorem 3.1. *The linear maps* $\{d'_k\}_{k\geq 2}$ *with* d'_1 *give an* A_∞*-category structure on the family of complexes* $\mathcal{C}' = \{V'(a,b)\}_{a,b\in \mathbf{Ob}(\mathcal{C})}$.

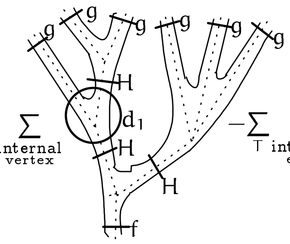
Proof.



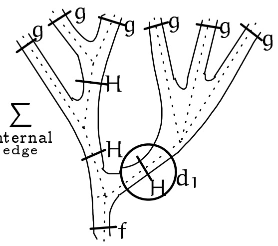
$= \sum_T$



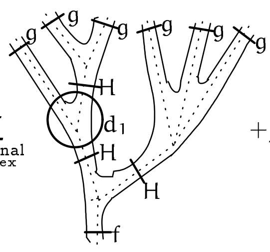
$= \sum_T \sum_{\substack{\text{internal}\\ \text{vertex}}}$



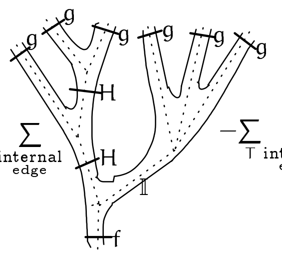
$- \sum_T \sum_{\substack{\text{internal}\\ \text{edge}}}$



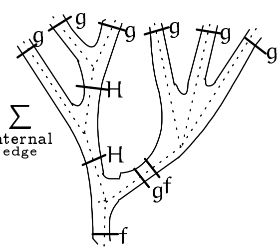
$= \sum_T \sum_{\substack{\text{internal}\\ \text{vertex}}}$



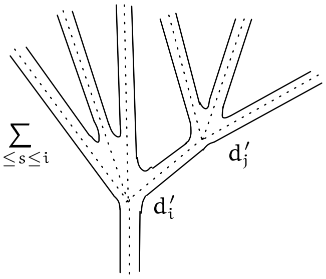
$+ \sum_T \sum_{\substack{\text{internal}\\ \text{edge}}}$



$- \sum_T \sum_{\substack{\text{internal}\\ \text{edge}}}$



$= - \sum_{\substack{i+j=k+1\\ i,j\geq 2}} \sum_{1\leq s\leq i}$



This is the A_∞relation (1.1) for $\{d'_k\}$. Here we use

- in the first line, the fact that f, g are chain maps;
- in the third line, the property (2.1) of the homotopy operator;
- in the fourth line, the cancellation of the first sum and the second sum because of the A_∞-relation (1.1) for $\{d_k\}$ ∎

Question 3.1. Can we give a natural 'geometric interpretation' of this construction?

References

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