From Geometry to Quantum Computation

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Abstract

The aim of this paper is to introduce our idea of Holonomic Quantum Computation (Computer). Our model is based on both harmonic oscillators and non–linear quantum optics, not on spins of usual quantum computation and our method is moreover completely geometrical.

We hope that therefore our model may be strong for decoherence.

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1 Introduction

Quantum Computation is a very attractive and challenging task in this century.
After the breakthrough by P. Shor [1] there has been remarkable progress in Quantum Computer or Computation (QC briefly). This discovery had a great influence on scientists. This drived not only theoreticians to finding other quantum algorithms, but also experimentalists to building quantum computers. See [2] and [3], [4] in outline.

On the other hand, Gauge Theories are widely recognized as the basis in quantum field theories. Therefore it is very natural to intend to include gauge theories in QC $\cdots$ a construction of "gauge theoretical" quantum computation or of "geometric" quantum computation in our terminology. The merit of geometric method of QC may be strong for the influence from the environment.

In [5] and [6] Zanardi and Rasetti proposed an attractive idea $\cdots$ Holonomic Quantum Computation (Computer) $\cdots$ using the non-abelian Berry phase (quantum holonomy in the mathematical language). See also [7] and [8] as another interesting geometric models.

In their model a Hamiltonian (including some parameters) must be degenerated because an adiabatic connection is introduced using this degeneracy [9]. In other words, a quantum computational bundle is introduced on some parameter space due to this degeneracy (see [5]) and the canonical connection of this bundle is just the above.

They gave a few simple but interesting examples to explain their idea. To make their works more mathematical and rigorous the author has given the mathematical reinforcement to their works, see [13], [14], [15] and [16]. But his works are still not sufficient.

In this talk we will introduce our Holonomic Quantum Computation and discuss some problems to be solved.

We strongly hope that young mathematical physicists will enter this attractive field.

2 Mathematical Preliminaries

We start with mathematical preliminaries. Let $\mathcal{H}$ be a separable Hilbert space over $\mathbf{C}$. For $m \in \mathbf{N}$, we set

$$St_m(\mathcal{H}) \equiv \left\{ V = (v_1, \cdots, v_m) \in \mathcal{H} \times \cdots \times \mathcal{H} | V^\dagger V = 1_m \right\} , \tag{1}$$

($V^\dagger V = 1_m \Longleftrightarrow \langle v_i | v_j \rangle = \delta_{ij}$) where 1_m is a unit matrix in $M(m, \mathbf{C})$. This is called a (universal) Stiefel manifold. Note that the unitary group $U(m)$ acts on $St_m(\mathcal{H})$ from the right:

$$St_m(\mathcal{H}) \times U(m) \rightarrow St_m(\mathcal{H}) \ : \ (V, a) \mapsto Va. \tag{2}$$

Next we define a (universal) Grassmann manifold

$$Gr_m(\mathcal{H}) \equiv \left\{ X \in M(\mathcal{H}) | X^2 = X, X^\dagger = X \text{ and } \mathrm{tr} X = m \right\} , \tag{3}$$

where $M(\mathcal{H})$ denotes a space of all bounded linear operators on $\mathcal{H}$. Then we have a projection

$$\pi : St_m(\mathcal{H}) \rightarrow Gr_m(\mathcal{H}) \ , \quad \pi(V) \equiv VV^\dagger \ , \tag{4}$$

compatible with the action (2) ($\pi(Va) = Va(Va)^\dagger = Vaa^\dagger V^\dagger = VV^\dagger = \pi(V)$).

Now the set

$$\{U(m), St_m(\mathcal{H}), \pi, Gr_m(\mathcal{H})\}\ , \tag{5}$$

is called a (universal) principal $U(m)$ bundle, see [10] and [13]. We set

$$E_m(\mathcal{H}) \equiv \{(X, v) \in Gr_m(\mathcal{H}) \times \mathcal{H} | Xv = v\}\ . \tag{6}$$

Then we have also a projection

$$\pi : E_m(\mathcal{H}) \rightarrow Gr_m(\mathcal{H})\ , \quad \pi((X, v)) \equiv X\ . \tag{7}$$

The set

$$\{\mathbf{C}^m, E_m(\mathcal{H}), \pi, Gr_m(\mathcal{H})\}\ , \tag{8}$$

is called a (universal) m-th vector bundle. This vector bundle is one associated with the principal $U(m)$ bundle (5) ..

Next let M be a finite or infinite dimensional differentiable manifold and the map $P : M \rightarrow Gr_m(\mathcal{H})$ be given (called a projector). Using this P we can make the bundles (5) and (8) pullback over M :

$$\left\{U(m), \widetilde{St}, \pi_{\widetilde{St}}, M\right\} \equiv P^* \{U(m), St_m(\mathcal{H}), \pi, Gr_m(\mathcal{H})\}\ , \tag{9}$$

$$\left\{\mathbf{C}^m, \widetilde{E}, \pi_{\widetilde{E}}, M\right\} \equiv P^* \{\mathbf{C}^m, E_m(\mathcal{H}), \pi, Gr_m(\mathcal{H})\}\ , \tag{10}$$

see [10]. (10) is of course a vector bundle associated with (9).

Let $\mathcal{M}$ be a parameter space and we denote by λ its element. Let $\lambda_{\mathbf{0}}$ be a fixed reference point of $\mathcal{M}$. Let H_λ be a family of Hamiltonians parameterized by $\mathcal{M}$ which act on a Fock space $\mathcal{H}$. We set $H_0 = H_{\lambda_{\mathbf{0}}}$ for simplicity and assume that this has a m-fold degenerate vacuum :

$$H_0 v_j = \mathbf{0}, \quad j = 1 \sim m. \tag{11}$$

These v_j's form a m-dimensional vector space. We may assume that $\langle v_i | v_j \rangle = \delta_{ij}$. Then $(v_1, \cdots, v_m) \in St_m(\mathcal{H})$ and

$$F_0 \equiv \left\{ \sum_{j=1}^{m} x_j v_j | x_j \in \mathbf{C} \right\} \cong \mathbf{C}^m.$$

Namely, F_0 is a vector space associated with o.n.basis $(v_1, \cdots, v_m)$.

Next we assume for simplicity that a family of unitary operators parameterized by $\mathcal{M}$

$$W : \mathcal{M} \rightarrow U(\mathcal{H}), \quad W(\lambda_{\mathbf{0}}) = \mathrm{id}. \tag{12}$$

is given and H_λ above is given by the following isospectral family

$$H_\lambda \equiv W(\lambda) H_0 W(\lambda)^{-1}. \tag{13}$$

In this case there is no level crossing of eigenvalues. Making use of $W(\lambda)$ we can define a projector

$$P : \mathcal{M} \rightarrow Gr_m(\mathcal{H}), \quad P(\lambda) \equiv W(\lambda) \left(\sum_{j=1}^{m} v_j v_j^\dagger \right) W(\lambda)^{-1} \tag{14}$$

and have the pullback bundles over $\mathcal{M}$

$$\left\{U(m), \widetilde{St}, \pi_{\widetilde{St}}, \mathcal{M}\right\}, \quad \left\{\mathbf{C}^m, \widetilde{E}, \pi_{\widetilde{E}}, \mathcal{M}\right\}. \tag{15}$$

For the latter we set

$$|vac\rangle = (v_1, \cdots, v_m). \tag{16}$$

In this case a canonical connection form $\mathcal{A}$ of $\left\{U(m), \widetilde{St}, \pi_{\widetilde{St}}, \mathcal{M}\right\}$ is given by

$$\mathcal{A} = \langle vac|W(\lambda)^{-1}dW(\lambda)|vac\rangle, \tag{17}$$

where d is a differential form on $\mathcal{M}$, and its curvature form by

$$\mathcal{F} \equiv d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}, \tag{18}$$

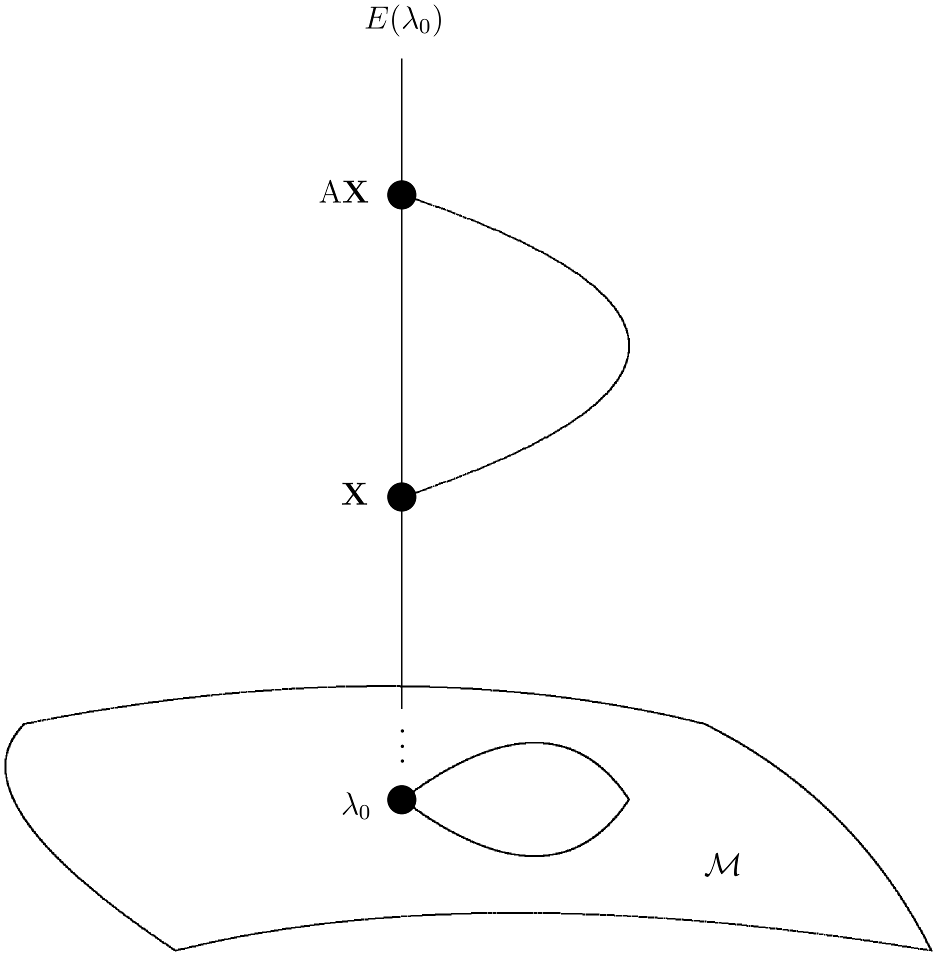
see [9] and [10].

Let γ be a loop in $\mathcal{M}$ at $\lambda_{\mathbf{0}}$., $\gamma : [0,1] \rightarrow \mathcal{M}, \gamma(0) = \gamma(1)$. For this γ a holonomy operator $\Gamma_{\mathcal{A}}$ is defined :

$$\Gamma_{\mathcal{A}}(\gamma) = \mathcal{P}exp\left\{\oint_{\gamma} \mathcal{A}\right\} \in U(m), \tag{19}$$

where $\mathcal{P}$ means path-ordered. This acts on the fiber F_0 at $\lambda_{\mathbf{0}}$ of the vector bundle $\left\{\mathbf{C}^m, \widetilde{E}, \pi_{\widetilde{E}}, M\right\}$ as follows : $\mathbf{x} \rightarrow \Gamma_{\mathcal{A}}(\gamma)\mathbf{x}$. The holonomy group $Hol(\mathcal{A})$ is in general subgroup of $U(m)$ .. In the case of $Hol(\mathcal{A}) = U(m)$, $\mathcal{A}$ is called irreducible, see [10].
In the Holonomic Quantum Computation we take

$$\begin{aligned} &\text{Encoding of Information} \Longrightarrow \mathbf{x} \in F_0, \\ &\text{Processing of Information} \Longrightarrow \Gamma_{\mathcal{A}}(\gamma) : \mathbf{x} \rightarrow \Gamma_{\mathcal{A}}(\gamma)\mathbf{x}. \end{aligned} \tag{20}$$



Quantum Computational Bundle

3 Holonomic Quantum Computation

We apply the results of last section to Quantum Optics and discuss (optical) Holonomic Quantum Computation proposed by [5] and [12]. Let $a(a^\dagger)$ be the annihilation (creation) operator of the harmonic oscillator. If we set $N \equiv a^\dagger a$ (: number operator), then

$$[N, a^\dagger] = a^\dagger \ , \ [N, a] = -a \ , \ [a, a^\dagger] = 1 \ . \tag{21}$$

Let $\mathcal{H}$ be a Fock space generated by a and $a^\dagger$, and $\{|n\rangle | n \in \mathbf{N} \cup \{0\}\}$ be its basis. The actions of a and $a^\dagger$ on $\mathcal{H}$ are given by

$$a|n\rangle = \sqrt{n}|n-1\rangle \ , \ a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \ , \tag{22}$$

where $|0\rangle$ is a vacuum $(a|0\rangle = 0)$. In the following we treat coherent operators and squeezed operators.

$$\text{Coherent Operator} \quad D(\alpha) = \exp\left(\alpha a^\dagger - \bar{\alpha} a\right) \quad \text{for } \alpha \in \mathbf{C}, \tag{23}$$

$$\text{Squeezed Operator} \quad S(\beta) = \exp\left(\beta \frac{1}{2}(a^\dagger)^2 - \bar{\beta}\frac{1}{2}a^2\right) \quad \text{for } \beta \in \mathbf{C}. \tag{24}$$

For the details see [13]. Next we consider the system of n–harmonic oscillators. If we set

$$a_i = 1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes \cdots \otimes 1, \quad {a_i}^\dagger = 1 \otimes \cdots \otimes 1 \otimes a^\dagger \otimes 1 \otimes \cdots \otimes 1, \tag{25}$$

for $1 \le i \le n$, then it is easy to see

$$[a_i, a_j] = [{a_i}^\dagger, {a_j}^\dagger] = 0,\ [a_i, {a_j}^\dagger] = \delta_{ij}. \tag{26}$$

We also denote by $N_i = {a_i}^\dagger a_i$ $(1 \le i \le n)$ number operators.

3.1 Two–Qubit Case

Since we want to consider coherent states based on Lie algebras $su(2)$ and $su(1,1)$, we make use of Schwinger's boson method, see [18] and [19]. Namely if we set

$$su(2): \quad J_+ = a_1^\dagger a_2,\ J_- = a_2^\dagger a_1,\ J_3 = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), \tag{27}$$

$$su(1,1): \quad K_+ = a_1^\dagger a_2^\dagger,\ K_- = a_2 a_1,\ K_3 = \frac{1}{2}\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right), \tag{28}$$

then we have

$$su(2): \quad [J_3, J_+] = J_+,\ [J_3, J_-] = -J_-,\ [J_+, J_-] = 2J_3, \tag{29}$$

$$su(1,1): \quad [K_3, K_+] = K_+,\ [K_3, K_-] = -K_-,\ [K_+, K_-] = -2K_3. \tag{30}$$

In the following we treat unitary coherent operators based on Lie algebras $su(2)$ and $su(1,1)$.

$$U(\lambda) = \exp\left(\lambda J_+ - \bar{\lambda} J_-\right) \quad \text{for } \lambda \in \mathbf{C}, \tag{31}$$

$$V(\mu) = \exp\left(\mu K_+ - \bar{\mu} K_-\right) \quad \text{for } \mu \in \mathbf{C}. \tag{32}$$

For the details of $U(\lambda)$ and $V(\mu)$ see [17] and [18].

Let H_0 be a Hamiltonian with nonlinear interaction produced by a Kerr medium., that is $H_0 = \hbar \mathrm{X} N(N-1)$, where X is a certain constant, see [11] and [12]. The eigenvectors of H_0 corresponding to 0 is $\{|0\rangle, |1\rangle\}$, so its eigenspace is $\mathrm{Vect}\{|0\rangle, |1\rangle\} \cong \mathbf{C}^2$. The space $\mathrm{Vect}\{|0\rangle, |1\rangle\}$ is called 1-qubit (quantum bit) space, see [2] or [3]. Since we are considering the system of two particles, the Hamiltonian that we treat in the following is

$$H_0 = \hbar \mathrm{X} N_1(N_1 - 1) + \hbar \mathrm{X} N_2(N_2 - 1). \tag{33}$$

The eigenspace of 0 of this Hamiltonian becomes therefore

$$F_0 = \mathrm{Vect}\{|0\rangle, |1\rangle\} \otimes \mathrm{Vect}\{|0\rangle, |1\rangle\} = \mathrm{Vect}\{|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle\} \cong \mathbf{C}^4. \tag{34}$$

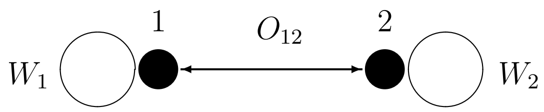
We set $|vac\rangle = (|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle)$. Next we consider the following isospectral family of H_0 :

$$H_{(\alpha_1,\beta_1,\lambda,\mu,\alpha_2,\beta_2)} = W(\alpha_1, \beta_1, \lambda, \mu, \alpha_2, \beta_2) H_0 W(\alpha_1, \beta_1, \lambda, \mu, \alpha_2, \beta_2)^{-1}, \tag{35}$$

$$W(\alpha_1, \beta_1, \lambda, \mu, \alpha_2, \beta_2) = W_1(\alpha_1, \beta_1) O_{12}(\lambda, \mu) W_2(\alpha_2, \beta_2). \tag{36}$$

where

$$O_{12}(\lambda, \mu) = U(\lambda)V(\mu), \quad W_j(\alpha_j, \beta_j) = D_j(\alpha_j) S_j(\beta_j) \quad \text{for} \quad j = 1, 2. \tag{37}$$



In this case

$$\mathcal{M} = \left\{ (\alpha_1, \beta_1, \lambda, \mu, \alpha_2, \beta_2) \in \mathbf{C}^6 \right\} \tag{38}$$

and we want to calculate

$$\mathcal{A} = \langle vac | W^{-1} dW | vac \rangle, \tag{39}$$

where

$$\begin{aligned} d = {} & d\alpha_1 \frac{\partial}{\partial \alpha_1} + d\bar{\alpha_1} \frac{\partial}{\partial \bar{\alpha_1}} + d\beta_1 \frac{\partial}{\partial \beta_1} + d\bar{\beta}_1 \frac{\partial}{\partial \bar{\beta}_1} + d\lambda \frac{\partial}{\partial \lambda} + d\bar{\lambda} \frac{\partial}{\partial \bar{\lambda}} + d\mu \frac{\partial}{\partial \mu} + d\bar{\mu} \frac{\partial}{\partial \bar{\mu}} \\ & + d\alpha_2 \frac{\partial}{\partial \alpha_2} + d\bar{\alpha_2} \frac{\partial}{\partial \bar{\alpha_2}} + d\beta_2 \frac{\partial}{\partial \beta_2} + d\bar{\beta}_2 \frac{\partial}{\partial \bar{\beta}_2} \ . \end{aligned} \tag{40}$$

The calculation of (39) is not easy, see [16] for the details.
Problem Is the connection form irreducible in $U(4)$?
Our analysis in [16] shows that the holonomy group generated by $\mathcal{A}$ may be $SU(4)$ not $U(4)$. To obtain $U(4)$ a sophisticated trick $\cdots$ higher dimensional holonomies [22] $\cdots$ may be necessary. A further study is needed.

3.2 N–Qubit Case

A reference Hamiltonian is in this case

$$H_0 = X \sum_{i=1}^{n} N_i (N_i - 1), \quad X \text{ is a constant}$$

and the eigen–space to 0–eigenvalue (n–qubits) becomes

$$F_0 = \mathrm{Vect}\{ |0, \cdots, 0, 0\rangle, |0, \cdots, 0, 1\rangle, \cdots, |1, \cdots, 1, 0\rangle, |1, \cdots, 1, 1\rangle \} \cong \mathbf{C}^{2^n} \tag{41}$$

and set $|vac\rangle = (|0, \cdots, 0, 0\rangle, |0, \cdots, 0, 1\rangle, \cdots, |1, \cdots, 1, 0\rangle, |1, \cdots, 1, 1\rangle)$.
The $u(n)$–algebra is defined by

$$\text{generators} \ : \ \{ E_{ij} \mid 1 \le i, j \le n \}, \quad \text{relations} \ : \ [E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}, \tag{42}$$

and $\{ E_{in} \mid 1 \le i \le n-1 \}$ a Weyl basis. Boson representation of $u(n)$–algebra is well–known to be

$$E_{ij} = {a_i}^{\dagger} a_j \quad 1 \le i, j \le n \ . \tag{43}$$

The $u(n-1, 1)$–algebra is also defined by

$$\text{generators} \ : \ \{ E_{ij} \mid 1 \le i, j \le n \}, \quad \text{relations} \ : \ [E_{ij}, E_{kl}] = \eta_{jk} E_{il} - \eta_{li} E_{kj}, \tag{44}$$

where $\eta = \mathrm{diag}(1, \cdots, 1, -1)$, and $\{E_{in} \mid 1 \le i \le n-1\}$ a Weyl basis. Boson representation of $u(n-1,1)$–algebra is given by

$$E_{ij} = {a_i}^\dagger a_j \quad 1 \le i, j \le n-1 \ , \quad E_{nn} = {a_n}^\dagger a_n + 1$$
$$E_{in} = {a_i}^\dagger {a_n}^\dagger \ , \quad E_{ni} = a_n a_i \quad 1 \le i \le n-1 \ . \tag{45}$$

A family of Hamiltonians that we treat is

$$H = W H_0 W^{-1} \ , \qquad W = \prod_{j=1}^{n} W_{jn} \tag{46}$$

and W_{jn} is

$$W_{jn}(\alpha_j, \beta_j, \lambda_j, \mu_j) = W_j(\alpha_j, \beta_j) O_{jn}(\lambda_j, \mu_j), \tag{47}$$

and

$$W_j(\alpha_j, \beta_j) = D_j(\alpha_j) S_j(\beta_j), \quad O_{jn}(\lambda_j, \mu_j) = U_j(\lambda_j) V_j(\mu_j), \tag{48}$$

and for $1 \le j \le n-1$

$$U_j(\lambda_j) = \exp(\lambda_j {a_j}^\dagger a_n - \bar{\lambda}_j {a_n}^\dagger a_j), \quad V_j(\mu_j) = \exp(\mu_j {a_j}^\dagger {a_n}^\dagger - \bar{\mu}_j a_n a_j) \tag{49}$$

making use of each Weyl basis. We note that $O_{nn} = 1$.

In this case

$$\mathcal{M} = \{(\cdots, \alpha_j, \beta_j, \lambda_j, \mu_j, \alpha_{j+1}, \beta_{j+1}, \cdots)\} \cong \mathbf{C}^{4n-2} \tag{50}$$

and

$$\mathcal{A} = \langle vac | W^{-1} dW | vac \rangle \ , \tag{51}$$

where

$$d = \sum_{j=1}^{n} d\alpha_j \frac{\partial}{\partial \alpha_j} + d\bar{\alpha}_j \frac{\partial}{\partial \bar{\alpha}_j} + d\beta_j \frac{\partial}{\partial \beta_j} + d\bar{\beta}_j \frac{\partial}{\partial \bar{\beta}_j}$$
$$+ \sum_{j=1}^{n-1} d\lambda_j \frac{\partial}{\partial \lambda_j} + d\bar{\lambda}_j \frac{\partial}{\partial \bar{\lambda}_j} + d\mu_j \frac{\partial}{\partial \mu_j} + d\bar{\mu}_j \frac{\partial}{\partial \bar{\mu}_j} \ . \tag{52}$$

We propose the following

Problem Is the connection form irreducible in $U(2^n)$?

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