POLARIZATION AND QUANTIZATION

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Abstract. In this article we explain a method of a geometric quantization, pairing of polarizations, and exhibit some of known results for the examples of such symplectic manifolds to which this method was applied to construct a quantization operator of geodesic flows, and discuss a further problem.

Contents

1. Introduction

In the most primitive sense of *quantization*, we mean a correspondence from functions on a phase space to operators acting on functions on a configuration space. Of course a configuration space is thought as a set of all states of a certain physical system and they will constitute a manifold in the mathematical sense and a phase space is its cotangent bundle with the natural symplectic structure. So, functions on the phase space are called classical observables and functions on the manifold are understood as representing quantum states. Such an interpretation of the correspondences can be extended between higher degree quantities like (co)homology classes of the manifold or its (co)tangent bundle and operator invariants ([Ar], [AS1], [AS2], [APS], [DG], [We],[Go1], [Fl], [Yo], these are not complete). In this paper we consider a problem of a geometric quantization under this understanding of the physical sense of quantization.

Now let M be a compact manifold. We denote by $\mathcal{F}^0(M)$ the group of invertible Fourier integral operators of order zero acting on functions (or more precisely half

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densities on M, and by $Diff^1_{sym}(T^*_0M)$ the group of homogeneous symplectomorphisms of the punctured cotangent bundle T^*_0M. Also by $\mathcal{P}^0(M)$ we denote the group of invertible pseudo-differential operators of order zero on M. We only consider the operators in $\mathcal{F}^0(M)$ having classical total symbols, that is, total symbols have asymptotic expansions by homogeneous functions of integer degrees on each local coordinate neighborhood. Then we have the exact sequence

$$0 \longrightarrow \mathcal{P}^0(M) \longrightarrow \mathcal{F}^0(M) \stackrel{\sigma}{\longrightarrow} Diff^1_{sym}(T^*_0M) \longrightarrow 0. \tag{1.1}$$

Here the map σ is the correspondence from a Fourier integral operator to its canonical relation. In our case canonical relations are the graphs of homogeneous symplectomorphisms.

There will be no natural ways to construct a section of this map σ on the "whole" group $Diff^1_{sym}(T^*_0M)$, however for examples, if we consider a subgroup consisting of the homogeneous symplectomorphisms coming from diffeomorphisms of the underlying manifold M, then we have the natural realization of each diffeomorphism as a Fourier integral operator and this gives us a section of the map σ restricted to this subgroup. Or if we take a compact subgroup in $Diff^1_{sym}(T^*_0M)$, then there is a section of the map σ realizing each symplectomorphism as an Hermite type Fourier integral operator (see [BG]). In these cases the section of the map σ is not unique.

To find a section for the map σ we consider the following way :

(1) Let $\mathfrak{h}$ be a Hilbert space consisting of certain class of "nice" functions defined on T^*_0M. If we have an operator $T : \mathfrak{h} \to L_2(M)$ which gives an isomorphism, then by taking a subgroup $G \subset Diff_{sym}(T^*_0M)$ which preserves $\mathfrak{h}$, then

(2) the problem is reduced to determine whether the operator $T \circ C \circ T^{-1}$ is a Fourier integral operator for $C \in G$ such that its canonical relation is the graph of C.

To construct such a subspace $\mathfrak{h}$ and an operator $T : \mathfrak{h} \to L_2(M)$ we employ a method developed by J. H. Rawnsley ([Ra1], [Ra2], [Wh]). We explain this in the next section § 2. There we put several strong assumptions A(1) $\sim$ A(5) on symplectic manifolds, then they give us quite explicit results. Although each assumption is satisfied by many symplectic manifolds, for the moment it is not easy to find symplectic manifolds which satisfy all the assumptions ([FT], [Ii1], [Ii2], [Sz1], [Sz2], [Sz3]). In § 3 we gather up the concrete results for several such symplectic manifolds (= punctured cotangent bundles of compact rank one symmetric spaces, [Ra2], [FY], [F1], [F2]).

In § 4 we describe precisely the space $\mathfrak{h}$ consisting of "nice" functions and an operator T for the punctured cotangent bundle of the quaternion projective space ([F2]).

2. Polarizations and their pairing

In this section we explain briefly a method, so called, pairing of polarizations for symplectic manifolds satisfying somewhat strong conditions, since in § 4 we only exhibit explicit calculations for a more special case.

Let $(X,\ \omega)$ be a symplectic manifold with the symplectic form ω. We assume that the cohomology class $[\omega]$ is an integer class, and denote by $(L,\ \nabla)$ the line bundle with the connection ∇ corresponding to the symplectic form ω. We say that a complex subbundle F in the complexified tangent bundle $TX \otimes \mathbb{C}$ is a *polarization*, if $\dim_{\mathbb{C}} F = 1/2 \dim\ X$ and satisfies *integrability* conditions. Among polarizations we only treat with real and complex polarizations, that is,

(a) F is real, if $F = \overline{F}$,
(b) G is complex, if $G \cap \overline{G} = \{0\}$.

We must still assume that a complex polarizations is positive, that is,

$$-\sqrt{-1}\omega(\xi,\ \overline{\xi}) \geq 0, \text{ for any } \xi \in\ G.$$

We know that if a symplectic manifold has a positive complex polarization, then it is a Kähler manifold, and the symplectic form can be a Kähler form. Also if F is a real polarization, we may think F is a complexification of a Lagrangian foliation.

Proposition 2.1. *Let F be a real polarization and G a positive complex polarization, then we have always*

$$F \cap \overline{G} = \{0\}. \tag{2.1}$$

Note that vectors in a positive complex polarization G are the anti-holomorphic tangent vectors.

Let F be a real polarization and G a positive complex polarization on X as above, then by the preceding proposition 2.1, we have

Proposition 2.2.

$$K^F \otimes \overline{K^G} \cong \bigwedge^{max} T^*X \otimes \mathbb{C}, \tag{2.2}$$

and $T^*X \otimes \mathbb{C}$ is trivialized by the *Liouville volume form*

$$\Omega_X = \frac{(-1)^{n(n-1)/2}}{n!}\omega^n,\ 2n = \dim X.$$

Hence we can define a *pairing* $< \alpha,\ \beta >$, for (smooth) sections $\alpha \in \Gamma(K^F, X)$ and $\beta \in \Gamma(K^G, X)$, by

$$\sqrt{-1}^n < \alpha,\ \beta > \Omega_X = \alpha \wedge \overline{\beta}. \tag{2.3}$$

In general, for a polarization F we denote the annihilator

$$F^\circ = \{\xi\ \in T^*M \otimes \mathbb{C} \mid\ < \xi,\ x >= 0, \text{ for any } x \in F\},$$

and by $K^F = \bigwedge^{max} F^\circ$, so called, the canonical line bundle of the polarization F ("max" means the highest degree exterior product). On the canonical line bundle we have a connection ∇^F *along* the polarization F. This is defined by the Lie derivative:

$$\nabla^F_\xi(s) = \mathbf{i}_\xi \circ ds,$$

where $\mathbf{i}_\xi$ is the interior product by a (complex)vector field $\xi \in \Gamma(F)$ taking values in F and $s \in \Gamma(K^F, X)$.

The pairing between K^F and K^G (F real, G positive complex) is defined in a natural way as above, but what we want to define is a pairing of their square root. In general, the existence of the square root $\sqrt{K^F}$ and $\sqrt{K^G}$ of canonical line bundles is guaranteed by the property that their Chern classes are divisible by 2 (in which case we call X being metaplectic). In our cases below, that is $X = T^*_0 P^n\mathbb{H}$ the punctured cotangent bundle of the quaternion projective space, there are nowhere vanishing smooth global section σ_F of K^F and a nowhere vanishing holomorphic global section σ_G of positive complex polarization K^G. So, we define simply the pairing of $\sqrt{\sigma_F}$ and $\sqrt{\sigma_G}$ as

$$< \sqrt{\sigma_F},\ \sqrt{\sigma_G} >= \sqrt{|< \sigma_F,\ \sigma_G >|}. \tag{2.4}$$

In Section 4 we present some results of explicit calculations of pairings, explicit constructions of a Hilbert space $\mathfrak{h}$ and operator T for the case X being the punctured cotangent bundle of the quaternion projective space. So From now on we work on only such symplectic manifolds (X, ω) like a punctured cotangent bundle, that satisfy further several strong conditions A(1) $\sim$ A(5):

- A(1) *The symplectic form is exact, i.e., $\omega = d\theta$ with a specific real one-form θ,*
- A(2) *there exists a real polarization F such that $< \theta,\ F >= 0$,*
- A(3) (i) *there exists a positive complex polarization G.*
  So we can see from the beginning that X is a complex (Kähler)manifold and G is a subbundle of $(0,1)$-vectors, and moreover assume that
  (ii) *there exists a $(1,0)$-form η such that $d\eta = \omega$ (note that $< \eta,\ G >= 0$).*
- A(4) *There exist*
  (i) *a nowhere vanishing smooth global section σ_F of K^F*
  and,
  (ii) *a nowhere vanishing holomorphic global section σ_G of the positive complex polarization G.*
- A(5) *Finally we assume that the leaf space X/F is a smooth metalinear (or more strongly orientable) manifold for the real polarization F.*

Under these assumptions we do not distinguish half forms and half densities. In fact from the assumption A(5) we can take a nowhere vanishing smooth global section σ_F as the pull back $\pi^*(\mathbf{d}vol)$ of a volume form $\mathbf{d}vol$ on the leaf space X/F (so A(4) (i) is superfluous). By A(1) our symplectic manifold can not be compact.

Since the connection ∇^F (resp. ∇^G) of K^F (resp. K^G) is defined as Lie derivative, we can introduce the connection on $\sqrt{K^F}$ (resp. $\sqrt{K^G}$) along F (resp. G). Hence we also have a connection on $L \otimes \sqrt{K^F}$ (resp. $L \otimes \sqrt{K^G}$) in an obvious way, and can define the space of smooth sections $\Gamma_F(L \otimes \sqrt{K^F},\ X)$ (resp. $\Gamma_G(L \otimes \sqrt{K^G},\ X)$)

which consist of parallel sections with respect to the connection on $L\otimes\sqrt{K^F}$ (resp. $L\otimes\sqrt{K^G}$).

The assumption A(1), $d\theta=\omega$ with a real one form θ, enables us to trivialize L together with a nowhere vanishing global section $\mathbf{s}_F$ and to introduce such an Hermitian inner product that satisfies $(\mathbf{s}_F,\mathbf{s}_F)_x=1$ at any point $x\in X$. So in terms of this section $\mathbf{s}_F$ the connection ∇ is expressed as

$$\nabla_\xi(\mathbf{s}_F)=2\pi\sqrt{-1}<\theta,\xi>\mathbf{s}_F.$$

Also by Assumption A(3) we can trivialize the line bundle L with such a global section $\mathbf{t}_G$ that the connection ∇ is now expressed as

$$\nabla_\xi(\mathbf{t}_G)=2\pi\sqrt{-1}<\eta,\xi>\mathbf{t}_G.$$

Let $\phi\cdot\mathbf{s}_F=\mathbf{t}_G$, then ϕ must satisfies

$$\xi(\phi)+2\pi\sqrt{-1}\cdot<\theta-\eta,\xi>\cdot\phi=0,\quad\text{for any vector field }\xi. \tag{2.5}$$

This has a unique solution except constant multiples.

Under these trivializations of line bundles the sections $\varphi\in\Gamma_G(L\otimes\sqrt{K^G},\ X)$ are written with holomorphic functions f on X as a form

$$\varphi=f\cdot\mathbf{t}_G\otimes\sqrt{\sigma_G},$$

and the sections $\psi\in\Gamma_F(L\otimes\sqrt{K^F},\ X)$ by

$$\psi=\pi^*(g)\cdot\mathbf{s}_F\otimes\sqrt{\mathbf{d}vol}$$

with $g\in C^\infty(X/F)$, $\pi:X\to X/F$.

Now we introduce inner products on $\Gamma_G(L\otimes\sqrt{K^G},\ X)$ and $\Gamma_F(L\otimes\sqrt{K^F},\ X)$ as follows: let $\varphi_i=f_i\cdot\mathbf{t}_G\otimes\sqrt{\sigma_G}\in\Gamma_G(L\otimes\sqrt{K^G},\ X)$ $(i=1,2)$, then

$$(\varphi_1,\varphi_2)_G\equiv\int_X f_1\cdot\overline{f_2}\cdot(\mathbf{t}_G,\mathbf{t}_G)\cdot\sqrt{|<\sigma_G,\sigma_G>|}\cdot\Omega, \tag{2.6}$$

where

$$\sqrt{-1}^{\,n}<\sigma_G,\sigma_G>\Omega=\sigma_G\wedge\overline{\sigma_G} \tag{2.7}$$

is defined by the same way as (2.3), and for $\psi_i=\pi^*(g_i)\cdot\mathbf{s}_F\otimes\sqrt{\pi^*(\mathbf{d}vol)}\in\Gamma_F(L\otimes\sqrt{K^F},\ X)$ $(i=1,2)$,

$$(\psi_1,\psi_2)_F\equiv\int_{X/F}g_1\cdot\overline{g_2}\cdot\mathbf{d}vol. \tag{2.8}$$

So the completion is just isomorphic to $L_2(X,\mathbf{d}vol)$.

Finally we define the pairing between $\Gamma_G(L\otimes\sqrt{K^G},\ X)$ and $\Gamma_F(L\otimes\sqrt{K^F},\ X)$ by

$$\langle\psi,\varphi\rangle\equiv\int_X g\cdot f\cdot(\mathbf{s}_F,\mathbf{t}_G)\cdot\sqrt{|<\sigma_F,\sigma_G>|}\cdot\Omega, \tag{2.9}$$

where $\psi=\pi^*(g)\cdot\mathbf{s}_F\otimes\sqrt{\pi^*(\mathbf{d}vol)}\in\Gamma_F(L\otimes\sqrt{K^F},\ X)$, $g\in C^\infty(X/F)$ and $\varphi=f\cdot\mathbf{t}_G\otimes\sqrt{\sigma_G}\in\Gamma_G(L\otimes\sqrt{K^G},\ X)$ (f : holomorphic on X).

Although there are many symplectic manifolds which satisfy only one of A(1), A(2) or A(3) like cotangent bundles or Kähler manifolds, it is not easy to find symplectic manifolds which satisfy all our assumptions and moreover at this stage we do not guarantee the existence of globally defined non-zero holomorphic functions on X. However for such symplectic manifolds, *our first problem, to construct a Hilbert space* $\mathfrak{h}$ *consisting of "nice" functions on* X *and an operator* $T : \mathfrak{h} \to L_2(X/F)$, *was reduced to calculate explicitly the quantities (2.4), (2.7), to solve the equation (2.5), and to show the non-degeneracy of the pairing (2.9).*

3. Compact rank one symmetric spaces

In this section we exhibit a positive complex polarization for the punctured cotangent bundles of the sphere, quaternion projective spaces and the Cayley projective plane, and (1.0)-forms η satisfying A(3) (ii) for each case. We also give a nowhere vanishing holomorphic global section for the cases of the sphere and the quaternion projective space. Of course complex projective plane satisfy all our assumptions A(1) $\sim$ A(5), but we omit it ([FT]). The sphere case plays an important role for describing the quaternion case. In all our cases, the real polarization is the vertical foliation of the projection map to the base space.

[**I**] Let S^n be a unit sphere in $\mathbb{R}^{n+1}$,

$$S^n = \{x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = \|x\|^2 = 1\},$$

with the standard Riemannian metric induced from the Euclidean space $\mathbb{R}^{n+1}$. We identify tangent and cotangent bundles by the standard metric:

$$T_0^*S^n \cong T_0S^n = \{(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|x\| = 1,\ y \neq 0,\ (x, y) = \sum x_i y_i = 0\}$$

It is well known ([Ra1], [Ra2], [So2]) of the Kähler structure on the punctured cotangent bundle and is realized by the map

$$\begin{aligned} \tau_S : T_0^*S^n = T_0S^n \ &\to \ \mathbb{C}^{n+1} \\ (x, y) &\mapsto \tau_S(x, y) = z = (z_0, \ldots, z_n), z_i = \|y\|\, x_i + \sqrt{-1} y_i. \end{aligned} \tag{3.1}$$

The image is characterized by an equation:

$$\tau_S(T_0S^n) = \{z \in \mathbb{C}^{n+1} \backslash \{0\} \mid \sum z_i^2 = 0\}. \tag{3.2}$$

The (1,0)-form η, $d\eta = \omega$, is given by

$$\eta = \sqrt{-2}\partial \|z\|,\ \|z\| = \sqrt{\sum |z_i|^2},$$

and we have a more precise relation:

$$\theta_{S^n} \ = \ \text{the canonical one form} \ = \ \frac{\sqrt{-1}}{\sqrt{2}}(\partial \|z\| - \overline{\partial} \|z\|). \tag{3.3}$$

A nowhere vanishing holomorphic global section σ_G of the canonical line bundle of this complex structure is given by

$$\sigma_G = \frac{2}{\|z\|^2}\sum_{j=0}^{n}(-1)^j \overline{z_j} dz_0 \wedge \cdots \wedge dz_n. \tag{3.4}$$

We will denote this by σ_S hereafter.

[**II**] We explain a Kähler structure on the punctured cotangent bundle of the quaternion projective space following [F1] precisely.

Let $\mathbb{H}$ be the quaternion number field over the real number field $\mathbb{R}$, which is generated by the basis $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ with the relations $\boldsymbol{e}_i\boldsymbol{e}_j$ given by the table

(3.5)

	$\boldsymbol{e}_0$	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$
$\boldsymbol{e}_0$	$\boldsymbol{e}_0$	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$
$\boldsymbol{e}_1$	$\boldsymbol{e}_1$	$-\boldsymbol{e}_0$	$\boldsymbol{e}_3$	$-\boldsymbol{e}_2$
$\boldsymbol{e}_2$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_3$	$-\boldsymbol{e}_0$	$\boldsymbol{e}_1$
$\boldsymbol{e}_3$	$\boldsymbol{e}_3$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1$	$-\boldsymbol{e}_0$

We regard $\mathbb{H}^n$ as a right $\mathbb{H}$-vector space with the $\mathbb{H}$-inner product

$$\langle h, k\rangle_{\mathbb{H}} = \sum_{i=1}^{n}\theta(h_i)k_i \tag{3.6}$$

where $h = (h_1, \dots, h_n), k = (k_1, \dots, k_n) \in \mathbb{H}^n$ and $\theta(x) = x_0\boldsymbol{e}_0 - x_1\boldsymbol{e}_1 - x_2\boldsymbol{e}_2 - x_3\boldsymbol{e}_3$ for $x = \sum_{i=0}^{3} x_i\boldsymbol{e}_i \in \mathbb{H}$. Then the $\mathbb{R}$ bilinear form $\langle h, k\rangle = \frac{1}{2}\{\langle h, k\rangle_{\mathbb{H}} + \langle k, h\rangle_{\mathbb{H}}\}$ on $\mathbb{H}^n$ defines a Euclidean inner product on $\mathbb{H}^n$ as a real vector space. We will denote its extension to the complexification $\mathbb{H}^n \otimes \mathbb{C}$ as a complex bilinear form with the same notation $\langle\,\cdot\,,\,\cdot\,\rangle$.

Let $M(n, \mathbb{H})$ be the space of $n \times n$ $\mathbb{H}$-matrices, and for $X = (x_{ij}) \in M(n, \mathbb{H})$ define respectively

$$\theta(X) = (\theta(x_{ij})), \tag{3.7}$$

$$({}^tX)_{ij} = x_{ji}, \tag{3.8}$$

$$\operatorname{tr} X = \sum x_{ii}, \tag{3.9}$$

$${}^TX = \theta({}^tX). \tag{3.10}$$

Each matrix $X \in M(n, \mathbb{H})$ defines a right $\mathbb{H}$-linear map $X : \mathbb{H}^n \to \mathbb{H}^n$, and we have

$$\langle Xh, k\rangle_{\mathbb{H}} = \left\langle h, {}^TXk\right\rangle_{\mathbb{H}}. \tag{3.11}$$

The group $Sp(n)$ is then defined as a group consisting of those matrices $X \in M(n, \mathbb{H})$ which preserve the $\mathbb{H}$-inner product.

The space $\mathcal{H}(n,\mathbb{H}) = \{X \in M(n,\mathbb{H}) | X = {}^{T}X\}$ is called a Jordan algebra with the Jordan product

$$X \circ Y = \frac{1}{2}(XY + YX). \tag{3.12}$$

Note that for $X \in \mathcal{H}(n,\mathbb{H})$, $\mathrm{tr}\, X \in \mathbb{R}\boldsymbol{e}_0 \cong \mathbb{R}$ and $\mathcal{H}(n,\mathbb{H})$ is equipped with a Euclidean inner product given by

$$\langle X, Y\rangle_{\mathbb{R}} = \mathrm{tr}(X \circ Y). \tag{3.13}$$

The inner product has the property:

$$\langle X \circ Y, Z\rangle_{\mathbb{R}} = \langle X, Y \circ Z\rangle_{\mathbb{R}}. \tag{3.14}$$

As is well-known, the quaternion projective space $P^n\mathbb{H}$ is the set of all $\mathbb{H}$-one-dimensional subspaces in $\mathbb{H}^{n+1}$. Here we identify $P^n\mathbb{H}$ with the subset in $\mathcal{H}(n+1,\mathbb{H})$:

$$\begin{aligned} P^n(\mathbb{H}) = \{P \in \mathcal{H}(n+1,\mathbb{H}) \mid P = (p_i\theta(p_j)), \\ p = (p_0, \ldots, p_n) \in \mathbb{H}^{n+1}, \langle p, p\rangle = 1\}. \end{aligned} \tag{3.15}$$

We consider the Riemannian metric on $P^n\mathbb{H}$ defined through the Hopf fibration $\pi: S^{4n+3} \to P^n\mathbb{H}$, where $S^{4n+3} = \{h \in \mathbb{H}^{n+1} \mid \langle h, h\rangle = \langle h, h\rangle_{\mathbb{H}} = 1\}$ has the standard metric.

Let us denote the isomorphism $\mathbb{H} \to M(2,\mathbb{C})$

$$\mathbb{H} \ni h \longmapsto \begin{pmatrix} x_0 + \sqrt{-1}x_1 & x_2 + \sqrt{-1}x_3 \\ -x_2 + \sqrt{-1}x_3 & x_0 - \sqrt{-1}x_1 \end{pmatrix} \in M(2,\mathbb{C}) \tag{3.16}$$

by ρ and we denote with the same notation ρ its complexification

$$\mathbb{H} \otimes \mathbb{C} \overset{\rho}{\overset{\sim}{\to}} M(2,\mathbb{C}). \tag{3.17}$$

Now we introduce the following spaces:

$$\mathbf{E}_S = \{(p, q) \in \mathbb{H}^{n+1} \times \mathbb{H}^{n+1} \mid \langle p, p\rangle = 1, \langle p, q\rangle = 0,\ q + p(q, p)_{\mathbb{H}} \neq 0\}.$$

The space $\mathbf{E}_S$ is $Sp(n+1)$-invariant and also invariant under the right action of $Sp(1)$.

$$\begin{aligned} \mathbf{E}^0_S &= \{(p, q) \in \mathbb{H}^{n+1} \times \mathbb{H}^{n+1} \mid \langle p, p\rangle = 1, q \neq 0, (q, p)_{\mathbb{H}} = 0\}, \\ \mathbf{E}_{\mathbb{H}} &= \left\{(P, Q) \mid P, Q \in \mathcal{H}(n+1,\mathbb{H}), \mathrm{tr}\, P = 1, P \circ Q = \frac{1}{2}Q, Q \neq 0\right\}, \\ \widetilde{\mathbf{E}_S} &= \left\{(B_0, \ldots, B_n) \in M(2,\mathbb{C})^{n+1} \mid z \wedge w \neq 0, \sum \det B_i = 0\right\}, \end{aligned}$$

where $B_i \in M(2,\mathbb{C})$ takes the form $B_i = \begin{pmatrix} z_{2i} & w_{2i} \\ z_{2i+1} & w_{2i+1} \end{pmatrix}$, and $z = (z_0, \dots, z_{2n+1})$, $w = (w_0, \dots, w_{2n+1})$.

$$\widetilde{\mathbf{E}_S^0} = \left\{ (B_0, \dots, B_n) \in \widetilde{\mathbf{E}_S} \mid \sum B_i B_i^* = -J \sum {}^t\overline{B_i} B_i J \right\},$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the condition $\sum B_i B_i^* = -J \sum {}^t\overline{B_i} B_i J$ can be rewritten as $\sum z_{2i}\overline{z}_{2i+1} + w_{2i}\overline{w}_{2i+1} = 0$.

$$\widetilde{\mathbf{E}_{\mathbb{H}}} = \left\{ A \in M(2n+2,\mathbb{C}) \mid \mathbb{J}A \ = \ {}^t A \mathbb{J}, \text{ rank } A = 2, \ A^2 = 0 \right\}.$$

Here we denote by $\mathbb{J}$

$$\mathbb{J} = \begin{pmatrix} J & & & O \\ & J & & \\ & & \ddots & \\ O & & & J \end{pmatrix} \in M(2n+2,\mathbb{C}).$$

We will define the norm of a matrix $A = (a_{ij}) \in M(n,\mathbb{C})$ by $\|A\| = \sqrt{\sum |a_{ij}|^2} = \sqrt{\mathrm{tr}(A\,A^*)}$ and the norm of an element $B \in \widetilde{\mathbf{E}}_S$ by $\|B\| = \sqrt{\sum \|B_i\|^2}$.

Remark 3.1. The inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{R}}$ on $\mathcal{H}(n+1,\mathbb{H})$ is the restriction of the $\mathbb{R}$-bilinear form $\dfrac{1}{2}\,\mathrm{tr}(X^T Y + Y^T X)$ on $M(n+1,\mathbb{H})$ and can be extended to $M(n+1,\mathbb{H}) \otimes \mathbb{C} \cong M(2n+2,\mathbb{C})$ as a complex bilinear form in a natural way. We will denote this bilinear form by

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{C}} : M(n+1,\mathbb{H}) \otimes \mathbb{C} \times M(n+1,\mathbb{H}) \otimes \mathbb{C} \to \mathbb{C}.$$

Then the Hermitian inner product on $M(n+1,\mathbb{H}) \otimes \mathbb{C} \cong M(2n+2,\mathbb{C})$ is given by $\left\langle A, \overline{B} \right\rangle_{\mathbb{C}}$ and the norm of $A \in M(2n+2,\mathbb{C}) \cong M(n+1,\mathbb{H}) \otimes \mathbb{C}$, which we introduced above, equals to $\|A\| = \sqrt{2\left\langle A, \overline{A} \right\rangle_{\mathbb{C}}}$, that is, $\left\langle A, \overline{A} \right\rangle_{\mathbb{C}} = 1/2 \cdot \mathrm{tr}(A\,A^*)$.

Next we define the maps $\alpha, \beta, \pi_{\mathbb{H}}, \pi_S, \tau_S$ and $\tau_{\mathbb{H}}$ among these spaces:

$$\begin{array}{rccccl} \alpha : & \mathbf{E}_S & \to & \mathbf{E}_{\mathbb{H}} & & \\ & \Cup & & \Cup & & \\ & (p,q) & \longmapsto & (P,Q), & & P = (p_i\theta(p_j)), Q = (p_i\theta(q_j) + q_i\theta(p_j)) \end{array}$$

$$\begin{array}{rccccl} \beta : & \widetilde{\mathbf{E}_S} & \to & \widetilde{\mathbf{E}_{\mathbb{H}}} & & \\ & \Cup & & \Cup & & \\ & (B_0, \dots, B_n) & \longmapsto & A = (A_{ij}), & & A_{ij} = -B_i J {}^t B_j J \end{array}$$

$$\begin{array}{rccc} \pi_S : & \mathbf{E}_S & \to & S^{4n+3} \\ & \Cup & & \Cup \\ & (p,q) & \longmapsto & p \end{array}$$

$$\begin{array}{rccc}
\pi_{\mathbb{H}}: & \mathbf{E}_{\mathbb{H}} & \to & P^n\mathbb{H} \\
& \Uplus & & \Uplus \\
& (P,Q) & \longmapsto & P
\end{array}$$

$$\begin{array}{rccl}
\tau_S: & \mathbf{E}_S & \to & \widetilde{\mathbf{E}_S} \\
& \Uplus & & \Uplus \\
& (p,q) & \longmapsto & (B_0,\ldots,B_n), \\
& \multicolumn{3}{l}{B_i = \rho(\|q\|\, p_i \otimes 1 + q_i \otimes \sqrt{-1}), \quad \|q\| = \sqrt{\langle q, q\rangle_{\mathbb{H}}}} \\
\tau_{\mathbb{H}}: & \mathbf{E}_{\mathbb{H}} & \to & \widetilde{\mathbf{E}_{\mathbb{H}}} \\
& \Uplus & & \Uplus \\
& (P,Q) & \longmapsto & A = (A_{ij}), \\
& \multicolumn{3}{l}{A = \|Q\|^2 (\rho(P_{ij})) - (\rho(Q_{ij}))^2 + \frac{\sqrt{-1}}{\sqrt{2}} \|Q\| (\rho(Q_{ij}))\,.}
\end{array}$$

Of course $\mathbf{E}_{\mathbb{H}}$ is identified with the punctured tangent bundle of $P^n\mathbb{H}$. The Riemannian metric $g_{\mathbb{H}}$ on $P^n\mathbb{H}$ defined through the Hopf fibration is given by

$$g_{\mathbb{H}}(Q_1, Q_2) = \frac{1}{2} < Q_1,\, Q_2 >_{\mathbb{R}} = \frac{1}{2}\, \mathrm{tr}(Q_1 \circ Q_2) \tag{3.18}$$

for (P, Q_1), $(P, Q_2) \in \mathbf{E}_{\mathbb{H}}$.

$\mathbf{E}_S$ is an open subspace of the tangent bundle of S^{4n+3}. It consists of those vectors which are not parallel to the fiber of the Hopf fibration.

The map τ_S(resp. $\tau_{\mathbb{H}}$) is an isomorphism between the spaces $\mathbf{E}_S$ and $\widetilde{\mathbf{E}_S}$ (resp. $\mathbf{E}_{\mathbb{H}}$ and $\widetilde{\mathbf{E}_{\mathbb{H}}}$). Also we have $\tau_S(\mathbf{E}^0_S) = \widetilde{\mathbf{E}^0_S}$ and

Proposition 3.2. (a) $\mathbf{E}^0_S$ *is* $Sp(1)$*-invariant, and* $\widetilde{\mathbf{E}^0_S}$ *is* $SU(2)$*-invariant,*

(b) *the following diagram is commutative:*

$$\begin{array}{ccccc}
S^{4n+3} & \xleftarrow{\pi_S} & \mathbf{E}^0_S & \xrightarrow{\tau_S} & \widetilde{\mathbf{E}^0_S} \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} \\
P^n\mathbb{H} & \xleftarrow[\pi_{\mathbb{H}}]{} & \mathbf{E}_{\mathbb{H}} & \xrightarrow{\tau_{\mathbb{H}}} & \widetilde{\mathbf{E}_{\mathbb{H}}}.
\end{array} \tag{3.19}$$

(c) *the map* τ_S *commutes (on* $\mathbf{E}_S$*) with the action of* $Sp(1) \overset{\rho}{\tilde{\to}} SU(2)$.

Remark 3.3. The maps $\beta \circ \tau_S$ and $\tau_{\mathbb{H}} \circ \alpha$ do not coincide on the whole space $\mathbf{E}_S$.

By the Riemannian metric we identify the tangent bundle of the quaternion projective space $P^n\mathbb{H}$ with their cotangent bundle as we did in [**I**] for the sphere. Both spaces $\mathbf{E}_S$ and $\mathbf{E}_{\mathbb{H}}$ can be seen as complex manifolds through the maps τ_S and $\tau_{\mathbb{H}}$. Then we have

Proposition 3.4. *Let $\omega_{\mathbb{H}}$ be the symplectic forms on the cotangent bundle of the quaternion projective space, then*

$$\omega_{\mathbb{H}} = \tau_{\mathbb{H}}^{*}\left(2^{\frac{1}{4}}\sqrt{-1}\,\overline{\partial}\partial\sqrt{\|A\|}\right). \tag{3.20}$$

In fact we have a more precise relation similar to (3.3):

Proposition 3.5.

$$\sqrt{-1}\tau_{\mathbb{H}}^{*}(\partial\sqrt{\|A\|} - \overline{\partial}\sqrt{\|A\|}) = 2^{\frac{3}{4}}\,\theta_{P^n\mathbb{H}} \tag{3.21}$$

where $\theta_{P^n\mathbb{H}}$ is the canonical one-form on the cotangent bundle $T^*P^n\mathbb{H}$.

When we consider the action of $SL(2,\mathbb{C}) = \rho(\{r \in \mathbb{H}\otimes\mathbb{C}\,|\,r\,\theta(r) = 1\})$ on $\widetilde{\mathbf{E}_S}$ from the right, we have

Proposition 3.6.

$$\beta:\ \widetilde{\mathbf{E}_S} \to \widetilde{\mathbf{E}_{\mathbb{H}}} \tag{3.22}$$

is a principal fiber bundle with the structure group $SL(2,\mathbb{C})$.

This principal bundle can be seen as a complexification of the Hopf fibration with "the complexified base space" (= the punctured cotangent bundle) and "the complexified structure group". We use this structure in describing a nowhere vanishing holomorphic global section of the canonical line bundle K^G of $T_0^*P^n\mathbb{H}$ in § 4.

[**III**] Finally we describe a positive complex polarization for the punctured cotangent bundle of the Cayley projective plane and give a $(1,0)$-form in Assumption A(3) (ii) ([F2]).

The Cayley number field $\mathcal{O}$ is a division algebra over $\mathbb{R}$ generated by the basis $\{\boldsymbol{e}_i\}_{i=0}^{7}$ including the quaternion number field $\mathbb{H}$. Especially, among the relations $\boldsymbol{e}_i\boldsymbol{e}_j$, we have

$$\boldsymbol{e}_1\boldsymbol{e}_4 = \boldsymbol{e}_5,\ \boldsymbol{e}_2\boldsymbol{e}_4 = \boldsymbol{e}_6 \text{ and } \boldsymbol{e}_3\boldsymbol{e}_4 = \boldsymbol{e}_7. \tag{3.23}$$

Hence $\mathcal{O}$ is identified with

$$\mathcal{O} \cong \mathbb{H}\oplus\mathbb{H}\boldsymbol{e}_4 \tag{3.24}$$

and the multiplication between $x = a + b\boldsymbol{e}_4$ and $y = h + k\boldsymbol{e}_4 \in \mathbb{H}\oplus\mathbb{H}\boldsymbol{e}_4$ is given by

$$x\cdot y = ah - \theta(k)b + \{ka + b\theta(h)\}\boldsymbol{e}_4, \tag{3.25}$$

where $h = \sum_{i=0}^{3} h_i\boldsymbol{e}_i \in \mathbb{H}$ ($h_i \in \mathbb{R}$) and $\theta(h) = h_0\boldsymbol{e}_0 - h_1\boldsymbol{e}_1 - h_2\boldsymbol{e}_2 - h_3\boldsymbol{e}_3$, and so on. We assume that the basis $\{\boldsymbol{e}_i\}_{i=0}^{7}$ are *orthonormal* and we will sometimes omit $\boldsymbol{e}_0$ (= identity element) and identify $\mathbb{R} = \mathbb{R}\boldsymbol{e}_0 \subset \mathbb{H} \subset \mathcal{O}$.

For $h = \sum_{i=0}^{7} h_i\boldsymbol{e}_i \in \mathcal{O}$, we denote

$$\theta(h) = h_0\boldsymbol{e}_0 - \sum_{i=1}^{7} h_i\boldsymbol{e}_i, \tag{3.26}$$

as for $h \in \mathbb{H}$.

Let $M(3,\mathcal{O})$ be the space of 3×3 matrices with entries in $\mathcal{O}$. By identifying $M(3,\mathcal{O})\cong M(3,\mathbb{R})\otimes_{\mathbb{R}}\mathcal{O}$, we denote for $X\in M(3,\mathcal{O})$

$$\theta(X)=X_0\otimes \boldsymbol{e}_0-\sum_{i=1}^{7}X_i\otimes \boldsymbol{e}_i, \tag{3.27}$$

$$^tX=\sum_{i=0}^{7}{}^tX_i\otimes \boldsymbol{e}_i \tag{3.28}$$

and

$$\operatorname{tr}X=\sum_{i=0}^{7}(\operatorname{tr}X_i)\boldsymbol{e}_i, \tag{3.29}$$

where $X=\sum_{i=0}^{7}X_i\otimes \boldsymbol{e}_i$, $X_i\in M(3,\mathbb{R})$.

Now consider the subspace $\mathfrak{J}$ in $M(3,\mathcal{O})$ defined by

$$\mathfrak{J}=\{X\in M(3,\mathcal{O})\mid \theta({}^tX)=X\}, \tag{3.30}$$

then $\dim_{\mathbb{R}}\mathfrak{J}=27$ and any $X\in\mathfrak{J}$ has the form

$$X=\begin{pmatrix}\xi_1\boldsymbol{e}_0 & x_3 & \theta(x_2)\\ \theta(x_3) & \xi_2\boldsymbol{e}_0 & x_1\\ x_2 & \theta(x_1) & \xi_3\boldsymbol{e}_0\end{pmatrix}$$

where $\xi_i\in\mathbb{R}$, $x_i\in\mathcal{O}$.

The space $\mathfrak{J}$ is called an exceptional Jordan algebra with the Jordan product

$$X\circ Y=\frac{1}{2}(XY+YX)\in\mathfrak{J}, \tag{3.31}$$

$X,Y\in\mathfrak{J}$.

$\mathfrak{J}$ has an inner product given by

$$\operatorname{tr}(X\circ Y)=(X,Y)\boldsymbol{e}_0. \tag{3.32}$$

In fact, for $X\in\mathfrak{J}$, we have $\operatorname{tr}X\in\mathbb{R}\boldsymbol{e}_0$, and

$$(X,Y)=\sum_{i=1}^{3}(\xi_i\eta_i+2\langle x_i,y_i\rangle) \tag{3.33}$$

where

$$X=\begin{pmatrix}\xi_1\boldsymbol{e}_0 & x_3 & \theta(x_2)\\ \theta(x_3) & \xi_2\boldsymbol{e}_0 & x_1\\ x_2 & \theta(x_1) & \xi_3\boldsymbol{e}_0\end{pmatrix}$$

$$Y=\begin{pmatrix}\eta_1\boldsymbol{e}_0 & y_3 & \theta(y_2)\\ \theta(y_3) & \eta_2\boldsymbol{e}_0 & y_1\\ y_2 & \theta(y_1) & \eta_3\boldsymbol{e}_0\end{pmatrix}$$

and $\langle x_i,y_i\rangle$ denote the inner product on $\mathcal{O}$.

Now the Cayley projective plane $P^2\mathcal{O}$ is defined as

Definition 3.7. $P^2\mathcal{O} = \{X \in \mathfrak{J} \mid X \circ X = X,\ \operatorname{tr} X = 1\}$.

The exceptional Lie group F_4 is defined as a group of algebra automorphisms of $\mathfrak{J}$, and acts on $P^2\mathcal{O}$ two-point homogeneously. So we have $P^2\mathcal{O} \cong F_4/\mathrm{Spin}(9)$, where $\mathrm{Spin}(9)$ is realized as a subgroup of F_4 consisting of those elements which leave $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{J}$ invariant.

The tangent bundle $TP^2\mathcal{O}$ is identified with the subset in $\mathfrak{J} \times \mathfrak{J}$ given by

$$TP^2\mathcal{O} = \left\{ (X, Y) \in \mathfrak{J} \times \mathfrak{J} \;\middle|\; X \circ X = X, \operatorname{tr} X = 1, X \circ Y = \frac{1}{2} Y \right\}. \tag{3.34}$$

We introduce the Riemannian metric on $P^2\mathcal{O}$ such a way that

$$(Y_1, Y_2)_{\mathbb{P}} = \frac{1}{2} \operatorname{tr}(Y_1 \circ Y_2) = \frac{1}{2}(Y_1, Y_2), \tag{3.35}$$

where (X, Y_1), $(X, Y_2) \in TP^2\mathcal{O}$.

Then from the inclusions $\mathbb{C} \subset \mathbb{H} \subset \mathcal{O}$, the complex projective plane $P^2\mathbb{C}$ and the quaternion projective plane $P^2\mathbb{H}$ are embedded isometrically into $P^2\mathcal{O}$ as totally geodesic submanifolds ([Be]).

In the following we identify the tangent bundle $TP^2\mathcal{O}$ and the cotangent bundle $T^*P^2\mathcal{O}$ through the metric above as usual. Under this identification the symplectic form $\omega_{\mathcal{O}}$ on $T^*P^2\mathcal{O}$ is given by

$$\omega_{\mathcal{O}} = -\frac{1}{2}(dX, dY), \tag{3.36}$$

where we should interpret the inner product (dX, dY) as a two-form in such a way that

$$-(dX, dY) = \sum_{i=1}^{3} d\eta_i \wedge d\xi_i + 2 \sum_{i=1}^{3} \sum_{\alpha=0}^{7} dy^i_\alpha \wedge dx^i_\alpha \tag{3.37}$$

restricted to $TP^2\mathcal{O}$, that is, we should regard that $\sum \eta_i \xi_i$ is replaced by $\sum d\eta_i \wedge d\xi_i$ and so on in the definition of the inner product on $\mathfrak{J}$

We can extend $\theta : \mathfrak{J} \longrightarrow \mathfrak{J}$, ${}^t : \mathfrak{J} \longrightarrow \mathfrak{J}, \operatorname{tr} : \mathfrak{J} \longrightarrow \mathcal{O}$, and the inner product $(\cdot, \cdot)$ to the complexification $\mathfrak{J} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{J}^{\mathbb{C}}$ in a natural way. So the Hermite inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{J}^{\mathbb{C}}$ is given by

$$\langle X, Y \rangle = (X, \overline{Y}), \tag{3.38}$$

where $\overline{X} = \sum_{\alpha=0}^{7} \overline{X}_\alpha \otimes \boldsymbol{e}_\alpha$, $X_\alpha \in M(3, \mathbb{C})$ and $\overline{X}_\alpha$ is the complex conjugate of X_α. The norm of these elements in $\mathfrak{J}$ and $\mathfrak{J}^{\mathbb{C}}$ is always written as $\|\cdot\|$, and we write the norm of the tangent vector $Y \in T_X(P^2\mathcal{O})$ by $\|Y\|_{\mathbb{P}}$.

Now consider the map $\tau_{\mathcal{O}} : T_0^* P^2\mathcal{O}(\cong T_0 P^2\mathcal{O}) \longrightarrow \mathfrak{J}^{\mathbb{C}}$ defined by

$$\begin{aligned}\tau_{\mathcal{O}}(X,Y) & \\ &= (\|Y\|^2 X - Y\circ Y)\otimes 1 + \frac{1}{\sqrt{2}}\|Y\|\, Y\otimes\sqrt{-1} \\ &= (2\,\|Y\|_{\mathbb{P}}^2 X - Y\circ Y)\otimes 1 + \|Y\|_{\mathbb{P}} Y\otimes\sqrt{-1},\end{aligned} \tag{3.39}$$

then we have

Theorem 3.8. *The map $\tau_{\mathcal{O}}$ gives an isomorphism between $T_0^* P^2\mathcal{O}$ and $\mathbb{E} = \{A \in \mathfrak{J}^{\mathbb{C}} \mid A\circ A = 0, A \neq 0\}$. Moreover*

$$\tau_{\mathcal{O}}^*(\sqrt{-1}\,\overline{\partial}\partial\,\|A\|^{\frac{1}{2}}) = \frac{1}{\sqrt{2}}\omega_{\mathcal{O}}. \tag{3.40}$$

In this case we have also a more precise relation similar to (3.3) and (3.21): Let $\theta_{P^2\mathcal{O}}$ be the canonical one form on $T^*P^2\mathcal{O}$, then

Proposition 3.9.

$$\sqrt{2}\theta_{P^2\mathcal{O}} \;=\; \sqrt{-1}\tau_{\mathcal{O}}^*(\partial\sqrt{\|A\|} - \overline{\partial}\sqrt{\|A\|}). \tag{3.41}$$

The two-form $\sqrt{-2}\,\overline{\partial}\partial\,\|A\|^{\frac{1}{2}}$ is itself a Kähler form on $\mathfrak{J}^{\mathbb{C}}\backslash\{0\}$, so that we can regard $\mathfrak{J}^{\mathbb{C}}\backslash\{0\}$ is a symplectic manifold. On this symplectic manifold the flow $\{\phi_t\}_{t\in\mathbb{R}}$ defined by

$$\phi_t : A \mapsto \phi_t(A) = e^{-2\sqrt{-1}t}\cdot A$$

is a Hamilton flow. The Hamiltonian of this flow is given by the function $f : A \mapsto \frac{1}{\sqrt{2}}\|A\|^{\frac{1}{2}}$. Since $\mathbb{E}$ is holomorphic and the flow $\{\phi_t\}$ leaves $\mathbb{E}$ invariant, the Hamiltonian of this flow on $\mathbb{E}$ is just the restriction of f to $\mathbb{E}$, that is, the Hamiltonian is the square root of the metric function. So the flow $\{\phi_t\}$ is the bicharacteristic flow of the square root of the Laplacian on $P^2\mathcal{O}$. Especially the flow restricted to the unit sphere $= \{(X,Y)\in TP^2\mathcal{O} : \|Y\|_{\mathbb{P}} = 1\}$ coincides with the geodesic flow. So we have

Corollary 3.10. *The geodesics $\gamma(t)$ on $P^2\mathcal{O}$ through a point X with the direction Y ($\|Y\|_{\mathbb{P}} = 1$ and $X\circ Y = \frac{1}{2}Y$) is given by*

$$\gamma(t) = \cos 2t\cdot(X - \frac{1}{2}Y\circ Y) + \frac{1}{2}\sin 2t\cdot Y + \frac{1}{2}Y\circ Y. \tag{3.42}$$

4. Quaternion projective spaces

In this section we describe a nowhere vanishing holomorphic global section σ_G of the canonical line bundle of the complex structure described in the preceding section for the punctured cotangent bundle of the quaternion projective space and give a explicit form of the pairing (2.9), together with the concrete values of pairings (2.4) and (2.7). We denote by $\sigma_{\mathbb{H}}$ instead of σ_G, since we have also that for the sphere case $= \sigma_S$, and by $\mathbf{dv}_{\mathbb{H}}$ instead of $\mathbf{d}vol$, the volume form of the Riemannian metric on $P^n\mathbb{H}$. So, we choose a nowhere vanishing global section σ_F of the canonical line bundle of the real polarization as $\sigma_F = \pi_{\mathbb{H}}^*(\mathbf{dv}_{\mathbb{H}})$, and similarly we take for the sphere case : $\sigma_F = \pi_S^*(\mathbf{dv}_S)$, $\mathbf{dv}_S$ is the Riemann volume form on S^{4n+3}.

Let Z be a vector field on $\underbrace{M(2,\mathbb{C}) \times \cdots \times M(2,\mathbb{C})}_{n+1} \setminus \{0\}$ defined by

$$Z = \frac{1}{\|B\|^2} \left(\sum_{i=0}^{2n+1} \overline{\frac{\partial D}{\partial z_i}} \frac{\partial}{\partial z_i} + \sum_{i=0}^{2n+1} \overline{\frac{\partial D}{\partial w_i}} \frac{\partial}{\partial w_i} \right) \tag{4.1}$$

where we denote $B = (B_0, \ldots, B_n) \in M(2,\mathbb{C}) \times \cdots \times M(2,\mathbb{C})$ and $B_i = \begin{pmatrix} z_{2i} & w_{2i} \\ z_{2i+1} & w_{2i+1} \end{pmatrix}$ as before and $D = \sum \det B_i$

We write the $(4n+3)$-form σ_S on $M(2,\mathbb{C}) \times \cdots \times M(2,\mathbb{C}) \setminus \{0\}$ (see 3.4) by

$$\sigma_S = \frac{1}{(2\sqrt{-1})^{2n+2}} \mathbf{i}_Z \left(dz_0 \wedge \cdots \wedge dz_{2n+1} \wedge dw_0 \wedge \cdots \wedge dw_{2n+1} \right). \tag{4.2}$$

Since $< dD,\ Z > \equiv\ 1$, we have

$$(2\sqrt{-1})^{2n+2} \cdot dD \wedge \sigma_S = dz_0 \wedge \cdots \wedge dz_{2n+1} \wedge dw_0 \wedge \cdots \wedge dw_{2n+1} \tag{4.3}$$

and the restriction of σ_S to $\sum \det B_i = 0$ is holomorphic and nowhere vanishing as explained in in § 2 [**I**]. We will denote the restriction of σ_S to $\widetilde{\mathbf{E}}_S$ with the same notation.

Let $\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$ be elements in $\mathfrak{su}(2) \subset \mathfrak{sl}(2,\mathbb{C})$. We denote by Y_1, Y_2 and Y_3 the vector fields on $\widetilde{\mathbf{E}}_S$ corresponding to these three elements respectively defined by the action of $SL(2,\mathbb{C})$.

Let us consider the $4n$-form σ on $\widetilde{\mathbf{E}}_S$ given by

$$\sigma = \mathbf{i}_{Y_3} \circ \mathbf{i}_{Y_2} \circ \mathbf{i}_{Y_1}(\sigma_S). \tag{4.4}$$

Then σ is a holomorphic $4n$-form, invariant under the action of the group $Sp(n+1)$ on $\widetilde{\mathbf{E}}_S$ from the left and $SL(2,\mathbb{C})$-invariant from the right. Also note that $det(Ad_g) = 1$ for any $g \in SL(2,\mathbb{C})$. So there exists a unique nowhere vanishing holomorphic $4n$-form $\sigma_{\mathbb{H}}$ on $\widetilde{\mathbf{E}}_{\mathbb{H}}$ such that

$$\beta^*(\sigma_{\mathbb{H}}) = \sigma. \tag{4.5}$$

This $4n$-form $\sigma_{\mathbb{H}}$ gives a holomorphic trivialization of the canonical line bundle $K_{\mathbb{H}}^{\mathcal{G}}$ of $\widetilde{\mathbf{E}}_{\mathbb{H}}$, and is $Sp(n+1)$-invariant.

Let V_1, V_2 and V_3 be three vector fields on the sphere S^{4n+3} corresponding to the elements $\boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_3$ in $\mathfrak{sp}(1) \subset \mathbb{H}$ defined through the action of $Sp(1)$ from the right.

We define three one-forms η_1, η_2 and η_3 on S^{4n+3} in such a way that

$$\eta_i(V_j) = \delta_{ij} \tag{4.6}$$

$$\eta_i(V) = 0 \quad \text{for any } V \in TS^{4n+3} \tag{4.7}$$

which is orthogonal to V_j $(j = 1, 2, 3)$.

Now we have the following relations among these vector fields, one-forms and volume elements:

$$\pi_*(\boldsymbol{dv}_S) = 2\pi^2 \boldsymbol{dv}_{\mathbb{H}}, \tag{4.8}$$

$$2\pi^2\pi^*(\boldsymbol{dv}_{\mathbb{H}}) = \mathbf{i}_{V_3} \circ \mathbf{i}_{V_2} \circ \mathbf{i}_{V_1}(\boldsymbol{dv}_S), \tag{4.9}$$

$$\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \mathbf{i}_{V_3} \circ \mathbf{i}_{V_2} \circ \mathbf{i}_{V_1}(\boldsymbol{dv}_S) = \boldsymbol{dv}_S. \tag{4.10}$$

Here π_* means the fiber integration of the Hopf bundle $\pi : S^{4n+3} \to P^n\mathbb{H}$.

Let $\theta_i = (\pi_S \circ \tau_S^{-1})^*\eta_i$. We decompose θ_i into

$$\theta_i = \theta_i' + \theta_i'' \tag{4.11}$$

with the holomorphic component θ_i' and the anti-holomorphic component θ_i'' in the complexified cotangent bundle $T^*(\widetilde{\mathbf{E}_S}) \otimes \mathbb{C}$.

Then, we have

Proposition 4.1.

$$\theta_1' \wedge \theta_2' \wedge \theta_3' \wedge (\mathbf{i}_{Y_3} \circ \mathbf{i}_{Y_2} \circ \mathbf{i}_{Y_1}(\sigma_S)) = \det(\theta_i'(Y_j))\sigma_S. \tag{4.12}$$

on $\widetilde{\mathbf{E}_S}$.

Put

$$\sigma_S \wedge \overline{\sigma}_S = A_S \Omega_S \tag{4.13}$$

and

$$\sigma_{\mathbb{H}} \wedge \overline{\sigma}_{\mathbb{H}} = A_{\mathbb{H}} \Omega_{\mathbb{H}} \tag{4.14}$$

where

$$\Omega_S = \frac{(-1)^{(4n+3)(2n+1)}}{(4n+3)!}\omega_S^{4n+3} = \frac{-1}{(4n+3)!}\omega_S^{4n+3} \tag{4.15}$$

and

$$\Omega_{\mathbb{H}} = \frac{(-1)^{2n(4n-1)}}{(4n)!}\omega_{\mathbb{H}}^{4n} = \frac{1}{(4n)!}\omega_{\mathbb{H}}^{4n} \tag{4.16}$$

are the Liouville volume forms on T^*S^{4n+3} and $T^*P^n\mathbb{H}$ respectively.

Then the invariance of σ_S and Ω_S under the transitive action of the group $SO(4n+4)$ on the unit sphere in the tangent bundle TS^{4n+3} and the invariance of $\sigma_{\mathbb{H}}$ and $\Omega_{\mathbb{H}}$ under the transitive action of $Sp(n+1)$ on the unit sphere in the tangent bundle $TP^n\mathbb{H}$ give

Proposition 4.2.

$$A_S = \boldsymbol{a}_S \|B\|^{4n+1} \tag{4.17}$$

$$A_{\mathbb{H}} = \boldsymbol{a}_{\mathbb{H}} \|A\|^{2n+2} \tag{4.18}$$

with two constants $\boldsymbol{a}_S$ *and* $\boldsymbol{a}_{\mathbb{H}}$.

Again by the same reasons as above, we can put

$$(\pi_S \circ \tau_S^{-1})^*(\boldsymbol{dv}_S) \wedge \overline{\sigma}_S = B_S \Omega_S, \quad B_S = \boldsymbol{b}_S \, \|B\|^{-1} \tag{4.19}$$

and

$$(\pi_{\mathbb{H}} \circ \tau_{\mathbb{H}}^{-1})^*(\boldsymbol{dv}_{\mathbb{H}}) \wedge \overline{\sigma}_{\mathbb{H}} = B_{\mathbb{H}} \Omega_{\mathbb{H}}, \quad B_{\mathbb{H}} = \boldsymbol{b}_{\mathbb{H}} \, \|A\|, \tag{4.20}$$

with two constants $\boldsymbol{b}_S$ and $\boldsymbol{b}_{\mathbb{H}}$. Then we have

Proposition 4.3.

$$\beta^*\left(\frac{A_{\mathbb{H}}}{B_{\mathbb{H}}}\right) = \det(\theta_i'(Y_j)) \frac{A_S}{B_S} \tag{4.21}$$

on $\widetilde{\mathbf{E}_S^0}$.

By (4.17) to (4.20) and $\|B\|^2 = \sqrt{2}\beta^*(\|A\|)$ on $\widetilde{\mathbf{E}_S^0}$ we have

Corollary 4.4.

$$2\pi^2 \frac{\boldsymbol{a}_S}{\boldsymbol{b}_S} \det(\theta_i'(Y_j)) = \left(\frac{1}{\sqrt{2}}\right)^{2n+1} \frac{\boldsymbol{a}_{\mathbb{H}}}{\boldsymbol{b}_{\mathbb{H}}}. \tag{4.22}$$

Hence $\det(\theta_i'(Y_j))$ *must be constant on* $\widetilde{\mathbf{E}_S^0}$.

Now we list the concrete values of these constants $\boldsymbol{a}_S$, $\boldsymbol{b}_S$, $\boldsymbol{a}_{\mathbb{H}}$, $\boldsymbol{b}_{\mathbb{H}}$ and $\det(\theta_i'(Y_j))$.

Proposition 4.5.

$$\boldsymbol{a}_S = -\sqrt{-1}, \tag{4.23}$$

$$\boldsymbol{b}_S = \sqrt{-1}, \tag{4.24}$$

$$\boldsymbol{a}_{\mathbb{H}} = 2^{n-2}, \tag{4.25}$$

$$\det(\theta_i'(Y_j)) = 2^{-3} = \det(\theta_i'(Y_j')) = \det(\theta_i''(Y_j'')) \text{ on } \widetilde{\mathbf{E}_S^0}, \tag{4.26}$$

$$\boldsymbol{b}_{\mathbb{H}} = -\frac{1}{\sqrt{2}\pi^2}. \tag{4.27}$$

Here Y_i' (resp. Y_i'') is the holomorphic (resp. anti-holomorphic) part of the vector field Y_i. Note that the value $\det(\theta_i'(Y_j))$ is not constant on all of $\widetilde{\mathbf{E}_S}$.

The calculations for the determination of these five constants are so tedious that we do not write down the details here. They are made by evaluating both sides of (4.13) (4.14) and (4.19) at a specific point. Especially to determine the constant $\boldsymbol{a}_{\mathbb{H}}$, we make use of the fiber bundle structure stated in Proposition 3.6 to consider a suitable local coordinate neighborhood in the space $\widetilde{\mathbf{E}_{\mathbb{H}}}$. Then we use the above Corollary 4.4 to determine the constant $\boldsymbol{b}_{\mathbb{H}}$.

Although it appears in the proof of Proposition 4.3([F2]), we emphasize the relation between the Liouville forms on T^*S^{4n+3} and $T^*P^n\mathbb{H}$:

Proposition 4.6.

$$\Omega_S = \frac{\beta^*(A_{\mathbb{H}})}{A_S}\frac{1}{|\det(\theta_i'(Y_j))|^2}\theta_1' \wedge \theta_2' \wedge \theta_3' \wedge \theta_1'' \wedge \theta_2'' \wedge \theta_3'' \wedge \beta^*(\Omega_{\mathbb{H}}). \tag{4.28}$$

Now we write down the pairing (2.9):

From the pairing (4.14) and by (4.18) we define the pairing of $\sqrt{\sigma_{\mathbb{H}}}$ by

$$< \sqrt{\sigma_{\mathbb{H}}}, \sqrt{\sigma_{\mathbb{H}}} >= \sqrt{|\boldsymbol{a}_{\mathbb{H}}|}\,\|A\|^{n+1}, \tag{4.29}$$

and hereafter we denote by $\boldsymbol{t}_{\mathbb{H}}$ instead of $\mathbf{t}_G$ (global section which trivializes L with respect to the complex polarization G), then the inner product (2.6) is expressed as

$$\begin{aligned}(\phi, \psi)_G &= \int_{\widetilde{\mathbf{E}_{\mathbb{H}}}} f(A)\overline{g(A)}\,(\boldsymbol{t}_{\mathbb{H}}, \boldsymbol{t}_{\mathbb{H}}) < \sqrt{\sigma_{\mathbb{H}}}, \sqrt{\sigma_{\mathbb{H}}} > \Omega_{\mathbb{H}} \\ &= \int_{\widetilde{\mathbf{E}_{\mathbb{H}}}} f(A)\overline{g(A)}\,e^{-2\sqrt[4]{2}\pi\sqrt{\|A\|}}|\boldsymbol{a}_{\mathbb{H}}|^{\frac{1}{2}}\,\|A\|^{n+1}\,\Omega_{\mathbb{H}},\end{aligned} \tag{4.30}$$

where $\phi = f\cdot\boldsymbol{t}_{\mathbb{H}}\otimes\sqrt{\sigma_{\mathbb{H}}}$, $\psi = g\cdot\boldsymbol{t}_{\mathbb{H}}\otimes\sqrt{\sigma_{\mathbb{H}}} \in \Gamma_G(L\otimes\sqrt{K_{\mathbb{H}}^G})$. Note that the determination of $(\boldsymbol{t}_{\mathbb{H}}, \boldsymbol{t}_{\mathbb{H}})$ is done by solving the equation (2.5) with the help of the relation (3.21), and is given by

$$\boldsymbol{t}_{\mathbb{H}} = e^{-2^{\frac{1}{4}}\pi\sqrt{\|A\|}}\cdot\boldsymbol{s}_{\mathbb{H}}.$$

We denote by $\mathfrak{h}_{\mathbb{H}}^G$ the completion of the space

$$\begin{aligned}&\{f\cdot\boldsymbol{t}_{\mathbb{H}}\otimes\sqrt{\sigma_{\mathbb{H}}}\,|\, f \text{ is a polynomial on } M(2n+2,\mathbb{C}) \text{ restricted to } \widetilde{\mathbf{E}_{\mathbb{H}}}\} \\ &\cong \sum_{l=0}^{\infty}\mathcal{P}_l^{\mathbb{H}}\end{aligned}$$

with respect to the norm defined by the inner product (4.30), where $\mathcal{P}_l^{\mathbb{H}}$ denotes the space of polynomials of degree l on $M(2n+2,\mathbb{C})$(restricted to $\widetilde{\mathbf{E}_{\mathbb{H}}}$). The spaces $\mathcal{P}_l^{\mathbb{H}}$ and $\mathcal{P}_{l'}^{\mathbb{H}}$ are orthogonal for $l \neq l'$ with respect to this inner product (4.30). This is our space $\mathfrak{h}$ we want to find out consisting of *nice* functions (= classical observables).

By (4.20) we define the pairing $< \sqrt{\pi_{\mathbb{H}}{}^*(\boldsymbol{dv}_{\mathbb{H}})}, \sqrt{\sigma_{\mathbb{H}}} >$ as

$$< \sqrt{\pi_{\mathbb{H}}{}^*(\boldsymbol{dv}_{\mathbb{H}})}, \sqrt{\sigma_{\mathbb{H}}} >= \sqrt{|\boldsymbol{b}_{\mathbb{H}}|}\,\|A\|^{\frac{1}{2}}, \tag{4.31}$$

then finally we define the pairing between the spaces $\Gamma_F(L\otimes\sqrt{K_{\mathbb{H}}^F})$ and $\Gamma_G(L\otimes\sqrt{K_{\mathbb{H}}^G})$ as follows: Let $\varphi = \pi_{\mathbb{H}}{}^*(f)\cdot\boldsymbol{s}_{\mathbb{H}}\otimes\sqrt{\pi_{\mathbb{H}}{}^*(\boldsymbol{dv}_{\mathbb{H}})}$ and $\psi = g\cdot\boldsymbol{t}_{\mathbb{H}}\otimes\sqrt{\sigma_{\mathbb{H}}}$, then we define

$$\begin{aligned}\langle\varphi, \psi\rangle &\equiv \int_{\widetilde{\mathbf{E}_{\mathbb{H}}}} \pi_{\mathbb{H}}{}^*(f)(A)\,g(A)\,(\boldsymbol{t}_{\mathbb{H}}, \boldsymbol{s}_{\mathbb{H}}) < \sqrt{\pi_{\mathbb{H}}{}^*(\boldsymbol{dv}_{\mathbb{H}})}, \sqrt{\sigma_{\mathbb{H}}} > \Omega_{\mathbb{H}} \\ &= \int_{\widetilde{\mathbf{E}_{\mathbb{H}}}} \pi_{\mathbb{H}}{}^*(f)(A)g(A)e^{-2^{\frac{1}{4}}\pi\sqrt{\|A\|}}\sqrt{|\boldsymbol{b}_{\mathbb{H}}|}\,\|A\|^{\frac{1}{2}}\,\Omega_{\mathbb{H}}.\end{aligned} \tag{4.32}$$

The pairing above defines an operator $T : g \mapsto T(g) \in C^\infty(P^n\mathbb{H})$ for $g \in C^\infty(\widetilde{\mathbf{E}_{\mathbb{H}}})$ satisfying a suitable integrability condition, that is we have

$$\int_{P^n\mathbb{H}} f\, T(g)\boldsymbol{dv}_{\mathbb{H}} = \langle \pi_{\mathbb{H}}^*(f)\cdot \boldsymbol{s}_{\mathbb{H}} \otimes \sqrt{\pi_{\mathbb{H}}{}^*(\boldsymbol{dv}_{\mathbb{H}})},\, g\cdot \boldsymbol{t}_{\mathbb{H}} \otimes \sqrt{\sigma_{\mathbb{H}}}\rangle.$$

In fact we have

$$T(g)\boldsymbol{dv}_{\mathbb{H}} = (\pi_{\mathbb{H}})_*(g e^{-2^{\frac{1}{4}}\pi\sqrt{\|A\|}}\sqrt{|\boldsymbol{b}_{\mathbb{H}}|}\,\|A\|^{\frac{1}{2}}\,\Omega_{\mathbb{H}}), \tag{4.33}$$

the fiber integration of the map $\pi_{\mathbb{H}}$.

Let $\Delta^{\mathbb{H}}$ be the Laplacian on $P^n\mathbb{H}$ with respect to the Riemannian metric defined in § 3 and let $\{e^{-\sqrt{-1}t\sqrt{\Delta^{\mathbb{H}}+(2n+1)^2}}\}_{t\in\mathbb{R}}$ be the one parameter family of unitary and elliptic Fourier integral operators generated by the square root of the operator $\Delta^{\mathbb{H}}+(2n+1)^2$.

The bicharacteristic flow $\{\ell_t\}_{t\in\mathbb{R}}$ of the operator $\sqrt{\Delta^{\mathbb{H}}+(2n+1)^2}$ is the Hamilton flow whose Hamiltonian is the square root of the metric function and can be expressed as follows:

Proposition 4.7.

$$\ell_t : \widetilde{\mathbf{E}_{\mathbb{H}}} \to \widetilde{\mathbf{E}_{\mathbb{H}}},\ \ell_t(A) = e^{-2\sqrt{-1}t}A. \tag{4.34}$$

The geodesic flow restricted to the unit tangent sphere bundle of $P^n\mathbb{H}$ coincides with the Hamilton flow above under the identification of the tangent bundle and the cotangent bundle by the Riemannian metric.

From the theory of Fourier integral operators ([Ho2]), we know that for each $t\in\mathbb{R}$ the graph of the symplectic isomorphism ℓ_t is the canonical relation of the Fourier integral operator $e^{-\sqrt{-1}t\sqrt{\Delta^{\mathbb{H}}+(2n+1)^2}}$. This correspondence can be seen as a quasi-classical approximation. An opposite correspondence is what we want to construct, and carried out by noting the followings: Since $\ell_t^*(\sigma_{\mathbb{H}}) = e^{-2\sqrt{-1}t(2n+1)}\sigma_{\mathbb{H}}$ and $\ell_t^*(\boldsymbol{t}_{\mathbb{H}}) = \boldsymbol{t}_{\mathbb{H}}$, we should regard the action of the flow $\{\ell_t\}$ on the Hilbert space $\mathfrak{h}_{\mathbb{H}}^{\mathcal{G}}$ such that $(\ell_t)^*(f\cdot \boldsymbol{t}_{\mathbb{H}}\otimes\sqrt{\sigma_{\mathbb{H}}}) = e^{-\sqrt{-1}t(2l+2n+1)}f\cdot\boldsymbol{t}_{\mathbb{H}}\otimes\sqrt{\sigma_{\mathbb{H}}}$ for each $f\in\mathcal{P}_l^{\mathbb{H}}$. Then we have

Theorem 4.8. (a) *The operator T is an isomorphism between the spaces $\mathfrak{h}_{\mathbb{H}}^G$ and $L_2(P^n\mathbb{H})$.*

(b) *The following diagram is commutative*

$$\begin{array}{ccc} \mathfrak{h}_{\mathbb{H}}^G & \xrightarrow{\ell_t^*} & \mathfrak{h}_{\mathbb{H}}^G \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ L_2(P^n\mathbb{H}) & \xrightarrow[e^{-t\sqrt{-1}\sqrt{\Delta^{\mathbb{H}}+(2n+1)^2}}]{} & L_2(P^n\mathbb{H}). \end{array} \tag{4.35}$$

5. Remarks

(a) To show that the operator T is an isomorphism, we need to determine certain fiber integrations in a quite explicit form in terms of the Γ-function ([F2]).

(b) There are two another ways to construct similar operators

$$T_i : \mathfrak{h}_{\mathbb{H}}^G \to L_2(P^n\mathbb{H}),\ \ i = 1, 2,$$

to the operator T above. These are constructed by making use of the corresponding operators constructed for the sphere ([Ra2]) and the complex projective space ([FY]), Hopf fibration and the complexified Hopf fibration (3.6). It might be desirable to know whether the operators $T_i \circ T^{-1}$ are pseudo-differential or not.

(c) Let $g \in Sp(n+1)$, then the (dual) differential of g,

$$(dg)^* : T_0^*P^n\mathbb{H} \to T_0^*P^n\mathbb{H}$$

is a holomorphic and of course symplectic transformation on $T_0^*P^n\mathbb{H}$ and leaves the function

$$e^{-2^{\frac{1}{4}}\pi\sqrt{\|A\|}}$$

invariant, which plays as a weight function in the fiber integration π_* for defining the operator T. These facts imply that the natural action (natural lift to $\mathcal{F}^0(P^n\mathbb{H})$) of g,

$$g^* : L_2(P^n\mathbb{H}, \mathbf{dv}_{\mathbb{H}}) \to L_2(P^n\mathbb{H}, \mathbf{dv}_{\mathbb{H}}),$$

coincides with the operator $T \circ d(g^{-1})^* \circ T^{-1}$.

(d) Each of our assumptions A(1) $\sim$ A(5) is not strong by itself, however it is not easy to find manifolds whose (punctured)cotangent bundle satisfies all assumptions simultaneously, and for the moment we know such examples as the cotangent bundle of the Euclidean space, and the punctured cotangent bundles of compact rank one symmetric spaces. In all these examples their geodesic flows are holomorphic transformations. In the papers [Sz1], [Sz2], [Sz3], [Ii1] and [Ii2] the authors constructed complex structures on (punctured) cotangent bundles of compact rank one symmetric spaces, which are not invariant under the action of geodesic flows and studied a relation between their complex structures and that we explained above.

For these symplectic manifolds it will be interesting to determine the group of holomorphic and symplectic (homogeneous) transformations. We only know completely for the case of the punctured cotangent bundle of spheres ([Ra1], also see [FT] for linear transformations).

(e) Here we did not exhibit a nowhere vanishing holomorphic global section of the canonical line bundle for the Cayley projective plane case. We will discuss this problem in a separate paper together with a construction of a quantization operator.

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