Deformation Quantization and Quantum mechanics

Akira Yoshioka*

25/09/2001, Waseda

Abstract

Results of existence, classification in formal deformation quantization are reviewed. A star product is introduced with a real number as a deformation parameter. An eigenvalue problem is discussed for convergent star product, with an expample of one dimensional harmonic oscillator.

1 Introduction

In a word, deformation Quantization is to introduce an associative noncommutative product into function spaces on manifolds. The product is given by deforming the trivial commutative pointwize multiplication of functions.

In this talk, first I will survey main results of **formal** deformation quantizations concerning the existence and the equivalence problems.

Then I will mension an observation on what can be done when the formal deformation parameter ν is replaced with a real number $\hbar$. I will show by using a star product a certain mechanical system on a phase space can be considered equivalent to some quantum system.

2 Formal deformation quantization

Let me start with showing the typical example, the *Moyal product*.

*e-mail: yoshioka@rs.kagu.sut.ac.jp. This work is supported by Grant-in-Aid for Scientific Research (C) (#13640088.), Japan Society for the Promotion of Science.

Poisson bracket On the 2-dimensional euclidean space $\mathbf{R}^2 \ni (x,y)$ we have a biderivation called the *Poisson bracket*

$$\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}, \quad f(x,y), g(x,y) \in C^\infty(\mathbf{R}^2) \tag{1}$$

Now using the over-left and over-right arrows, we write the bracket as

$$\{f,g\} = f\left(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y - \overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x\right) g = f\,\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\, g \tag{2}$$

Here we remark the left arrow over partial derivative means the partial derivative of the left function f, and so-on. With this notation we define bidifferential operaotor by

$$\begin{aligned}\left(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right)^2 &= \left(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y - \overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x\right)^2 \\ &= \left(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y\right)^2 - 2\left(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y\right)\cdot\left(\overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x\right) \\ &+ \left(\overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x\right)^2 \\ &= \overleftarrow{\partial}_x^2 \cdot \overrightarrow{\partial}_y^2 - 2\overleftarrow{\partial}_x\overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_y\overrightarrow{\partial}_x + \overleftarrow{\partial}_y^2 \cdot \overrightarrow{\partial}_x^2 \end{aligned} \tag{3}$$

Similarly, by using the binomial theorem we obtain bidifferential operators

$$\left(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \overleftarrow{\partial}_x^{n-k}\overleftarrow{\partial}_y^{k} \cdot \overrightarrow{\partial}_x^{k}\overrightarrow{\partial}_y^{n-k} \tag{4}$$

$(n = 3, 4, \cdots)$.

Moyal product $*_0$ Now, we introduce a formal parameter ν. For smooth functions $f(x,y), g(x,y) \in C^\infty(\mathbf{R}^2)$, we consider a formal power series

$$\begin{aligned} f *_0 g &= f \exp\frac{\nu}{2}\left(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right) g \\ &= f \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\nu}{2}\right)^n \left(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right)^n g \quad \in C^\infty(\mathbf{R}^2)[[\nu]] \end{aligned} \tag{5}$$

This operation can be taken for elements of $C^\infty(\mathbf{R}^2)[[\nu]]$ in the obvious way.

The product $*_0$ on $C^\infty(\mathbf{R}^2)[[\nu]]$ has the following property.

(D1). For functions $f(x,y), g(x,y) \in C^\infty(\mathbf{R}^2)$ the product is expanded as

$$f *_0 g = fg + \frac{\nu}{2}\{f,g\} + \nu^2\,\pi_2(f,g) + \cdots + \nu^n\,\pi_n(f,g) + \cdots$$

This means the product is a deformation of the trivial multiplication of functions and the first order term of ν is the Poisson brakcket.

(D2). The product is associative.

(D3). Each π_n is a bidifferential operator for $n = 2, 3, \cdots$.

Remark that on $\mathbf{R}^{2n} \ni (x_1, \cdots, x_n, y_1, \cdots, y_n)$ the product $*_0$ can be defined similarly by the Poisson bracket

$$\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y = \sum_{j=1}^{n} \left(\overleftarrow{\partial}_{x_j} \cdot \overrightarrow{\partial}_{y_j} - \overleftarrow{\partial}_{y_j} \cdot \overrightarrow{\partial}_{x_j} \right)$$

Further, this concept is also extended on the general Poisson manifolds.

Deformation quantization A skew biderivation on a manifold M is expressed locally as

$$\{f, g\} = \sum_{i,j} a_{ij}(x) \partial_{x_i} f \partial_{x_j} g$$

for $f, g \in C^\infty(M)$. A skew derivation $\{\ ,\ \}$ satisfiying the Jacobi identity is called a *Poisson bracket.*

Definition 1 *An* $\mathbf{R}[[\nu]]$*-bilinear (or* $\mathbf{C}(M)[\nu]]$*-bilinear) product* $*$ *on the function space* $C^\infty(M)[[\nu]]$ *is called* **star product** *when it satisfies the properties (D1)–(D3). The algebra* $(\mathbf{C}(M)[\nu]], *)$ *is called a* **Deformation quantization**.

The background One sees the commutator of the Moyal product $*_0$ satisfies

$$[x, y] = x *_0 y - y *_0 x = xy + \frac{\nu}{2} - (xy - \frac{\nu}{2}) = \nu$$

the Heisenberg's canonical commutation relation. Then the algebra of polynomilas $(\mathcal{P}[x, y, \nu], *_0)$ can be regarded as the algebra of operators generated by the derivation ∂_x and the multiplication x, and the algebra $(C^\infty(\mathbf{R}^2)[[\nu]], *)$ is a completion under certain topology. Motivated by the consideration above, deformation quantization was initiated in order to generalize the quantization from on euclidean space to on manifolds ([BFFLS]).

Main results

1. Existence On any Poisson manifold $(M, \{\ ,\ \})$, there exists a star product. The existence problem was solved for a symplectic manifold by [DL], [F], [OMY] and for a Poisson manifold by [K].

2. Classification Two star products $*$, $*'$ are called *equivalent* iff there exists an $\mathbf{R}[[\nu]]$-linear (or $\mathbf{C}[[\nu]]$-linear) isomorphism

$$T : C^\infty(M)[[\nu]] \to C^\infty(M)[[\nu]],$$

$$T = I + \nu T_1 + \nu^2 T_2 + \cdots + \nu^k T_k + \cdots , \quad T_k \text{ (differential operators)}$$

such that

$$f *' g = T^{-1}(Tf * Tg), \quad f, g \in C^\infty(M)[[\nu]].$$

The equivalence classes of star products $\{*\}/\sim$ are known; for a symplectic M, we have

$$\{*\}/\sim = H^2(M)[[\nu]]$$

([BCG], [F], [OMMY]), and for a Poisson M, the equivalence classes are isomorphic to the space of formal Poisson brackets [K].

3. Quantized Darboux theorem For a symplectic manifold M, a star product $*$ is locally equivalent to the Moyal product, i.e, for every point $p \in M$, there exists a canonical coordinate neighbourhood U where products $*$ and $*_0$ are equivalent ([G], [L]).

4. Geometric picture When M is symplectic, a star product is given in the geometric way as follows ([OMY]).

1. There exist a fiber bundle with Weyl algebra fiber $(W, \widehat{*})$

$$\pi : W_M \to M$$

2. There is an subalgebra $\mathcal{F}(W_M)$ of smooth sections of the bundle W_M, (algebra structure of sections are given by the pointwise multiplication of W_M), and also an algebra isomorphism

$$\sigma : (C^\infty(M)[[\nu]], *) \to (\mathcal{F}(W_M), \widehat{*}) \tag{6}$$

The algebra $\mathcal{F}(W_M)$ is given by a null space of a certain flat connection of W_M, so-called a Fedosov connection ([F], [X]). Further, there exists a contact algebra bundle $\mathcal{C}_{\mathcal{M}}$ containing W_M as a subbundle and there exists a non-flat connection whose restriction to W_M is the previous connection ([Y1], [Y2], [Y3]).

3 Star products with a real parameter $\hbar$

If we substitute a number into the parameter ν, the Moyal product $f *_0 g$ diverges for functions $f, g \in C^\infty(\mathbf{R}^2)$ in general. The main results up to now are based on the fact that ν is a formal parameter, and the most techniques are not useful for a numerical deformation paremater $\hbar$. We have no general theory of convergent star products at present, but, on the other hand we have some interesting obervation as follows.

Vacuum Let us put $\nu = i\hbar$, $\hbar > 0$ and let us consider the Moyal product $*_0$ on $\mathbf{R}^2$. In the sequal we deonte simply by $*_0 = *$.

Now we can consider a function, called a *vaccum*

$$v_0 = \exp\left(\frac{2}{\hbar i} xy\right)$$

since $\hbar$ is a number. An easy calculation gives

$$y * v_0 = y \exp \frac{\nu}{2}\left(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right) v_0 = y v_0 + \frac{i\hbar}{2} y(-\overleftarrow{\partial}_y)(\overrightarrow{\partial}_x) v_0 = 0. \qquad (7)$$

$$x * v_0 = 2x v_0. \qquad (8)$$

Obvious identities

$$x * x = x^2, \quad \underbrace{x * \cdots * x}_{n} = x^n$$

yield that a polynomial $p(x)$ satisfies

$$\begin{aligned} y * p(x) * v_0 &= p(x) * y * v_0 - [p(x), y] * v_0 = [y, p(x)] * v_0 \\ &= \tfrac{\hbar}{i} \partial_x p(x) * v_0 \end{aligned}$$

$$x * p(x) * v_0 = x p(x) * v_0$$

Thus, on the set $\mathcal{P} = \{p(x) * v_0 \,|\, p(x) \text{ polynomials}\}$ the following star product operations are represented as derivation and multiplication operators

$$y * \rightsquigarrow \tfrac{\hbar}{i} \partial_x, \quad x * \rightsquigarrow x \times \qquad (9)$$

The one dimensional harmonic oscillator Using the deformation parameter $\hbar$, we can handle a system with the Moyal product $*_0$ which seems equivalet to the one dimensional quantum harmonic oscillator.

We set as follows.

1. We identitfy $\mathbf{R}^2 = \mathbf{C}$ via $z = y + ix$.
2. We consider the hamiltonian funciton $H = \frac{1}{2}(x^2 + y^2) = \frac{1}{2} z\bar{z}$.
3. We set an element $u_0 = \exp\left(-\frac{1}{\hbar}(x^2 + y^2)\right) = \exp\left(-\frac{1}{\hbar} z\bar{z}\right)$ called the *vacuum*.
4. We introduce an inner product by $(f, g) = \frac{1}{\pi\hbar} \iint_{\mathbf{R}^2} f * \bar{g}\, dxdy$ and we consider a function space by
$$\mathcal{H} = \{f(z) * u_0 \,|\, f(z) = \sum_n a_n z^n,\ \sum_n |a_n|^2 \pi (2\hbar)^n n! < \infty\}$$
Then, $(\mathcal{H}, (\ ,\ ))$ is a Hilbert space.
5. A calcuation with (5) shows the functions $A = (1/\sqrt{2\hbar})\bar{z}$, $A^\dagger = (1/\sqrt{2\hbar})z$ satisfy
$$A * u_0 = 0, \quad A^\dagger * u_0 = (1/\sqrt{2\hbar}) z u_0.$$
$$[A, A^\dagger] = 1, \quad [A, H] = \hbar A, \quad [A^\dagger, H] = -\hbar A^\dagger,$$
$$H = \hbar\left(A^\dagger * A + \frac{1}{2}\right) = \hbar\left(A * A^\dagger - \frac{1}{2}\right)$$
Then, one sees on the space $\mathcal{H}$ the $*_0$ multiplication of functions A and $A^\dagger$ are represented as a dervation and a multiplication such as
$$A \rightsquigarrow \sqrt{2\hbar}\partial_z, \quad A^\dagger \rightsquigarrow \tfrac{1}{\sqrt{2\hbar}} \times$$
The identities shows
$$H * u_0 = \tfrac{\hbar}{2} u_0, \quad H * \left((A^\dagger)^n * u_0\right) = \hbar(n + \tfrac{1}{2})(A^\dagger)^n * u_0$$
H is an operator on $\mathcal{H}$ and $\hbar(n + \frac{1}{2})$, $n = 0, 1, \cdots$ are eigenvalues. Put $C_n = \|u_n\|$, where $u_n = \left(A^\dagger\right)^n * u_0$. Then $v_n = u_n / C_n$, $n = 0, 1, \cdots$ form a complete orthnormal system of $\mathcal{H}$. Hence, one sees the spectrum is
$$\{\hbar(n + \tfrac{1}{2}) \,|\, n = 0, 1, 2, \cdots\}$$
This is the same as the spectrum of the Schrodinger operator for one dimensional harmonic oscillator.

Remark The differentials at $\hbar = 0$ vanish for $u_0 = \exp\left(-\frac{1}{\hbar}z\bar{z}\right)$. Then, u_0 becomes 0 when regarded as a formal pawer seires of $\hbar$. Thus, the argument above can not be set in the formal deformation quantization theory.

Conclusion If we introduce a real number $\hbar$ as a deformation parameter, we can deal with an eigenvalue problem in the deformation quantization theory similarly as in the operator quantum mechanics. However, most techniques of formal deformation quantization thoery are not useful for convergent star products, hence we have to seek new techniques in deformation quantization with a number parameter.

References

[BCG] M. Bertelson, M. Cahen, and S. Gutt. *Equivalence of deformation quantization*, Classical quantum gravity 14 (1997), 157–162.

[BFFLS] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowizc, and D. Sternheimer, *Deformation theory and quantization*, Ann. Physics **111** (1978), 61-111.

[DL] M. De Wilde and P. B. A. Lecomte, *Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds*, Lett. Math. phys. **7** (1983), 487-496.

[F] B. Fedosov, *A simple geometrical construction of deformation quantization*, J. Differential Geom. **40** (1994), 213-238.

[G] S. Gutt, *Cohomology associated with the Poisson Lie algebra*, Lett. Math. Phys. **3** (1979), 297–310.

[K] M. Kontsevich, *Deformation quantization of Poisson manifolds I*, q-alg/9709040.

[L] A. Lichnerowizc, *Déformaions d'algèbres associèes à une variété symplectique (les $*_\nu$-products)*, Ann. Inst. Fourier, Grenoble **32** (1982), 157–209.

[OMY] H. Omori, Y. Maeda and A. Yoshioka, *Weyl manifolds and Deformation quantization*, Adv. Math. **85**(1991), 224-255.

[OMMY] H. Omori, Y. Maeda,N. Miyazaki and A. Yoshioka: *Poincaré-Cartan class and deformation quantization of Käler manifolds*, *Comm. Math. Phys.* **194** (1998), 207–230.

[X] X. Ping: *Fedosov $*$-products and quantum momentum maps. Comm. Math. Phys.* **197** (1998), 167–197.

[Y1] A. Yoshioka, *Contact structures on Weyl manifolds and Deformation quantization*, Proceedings of the conference on "Dynamical systems and differential geometry", RIMS Kokyuroku **1119**(1999), 1-17. (in Japanese).

[Y2] A. Yoshioka, *Weyl manifold and quantized connection*, Proceedings of the Workshop "Towards 100 years after Sophus Lie "(Kazan, 1998), Lobachevskii J. Math. **4** (1999), 177–206 (electronic).

[Y3] A. Yoshioka, *Contact Weyl manifold over a symplectic manifold* (to appear)